$\vec{\mathcal{C}}$-Homogeneous Graphs Via Ordered Pencils

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Abstract

Let $\mathcal{C}$ be a class of graphs and let $\vec{\mathcal{C}}$ be obtained from $\mathcal{C}$ by arc anchorage. A concept of $\vec{\mathcal{C}}$-homogeneous graphs that include the $\mathcal{C}$-ultrahomogeneous graphs is given, with the explicit construction of $\vec{\mathcal{C}}$-homogeneous graphs that are not $\mathcal{C}$-ultrahomogeneous, via ordered pencils of binary projective spaces.

1 Introduction

Let G be a finite, undirected, simple graph, and let W_1, W_2 be vertex subsets of G. In Gardiner's paper [4], G is *homogeneous* (resp. *ultrahomogeneous*) if whenever the induced subgraphs $G[W_1], G[W_2]$ are isomorphic, some isomorphism (resp. every isomorphism) of $G[W_1]$ onto $G[W_2]$ extends to an automorphism of G. Moreover, Gardiner gave an explicit characterization of the ultrahomogeneous graphs, using previous work of Sheehan [9]. In [8], Ronse showed that every homogeneous graph is ultrahomogeneous. See also [6, 1].

Let $\mathcal{C}$ be a class of graphs. In [7], Isaksen et al. declare a graph G to be $\mathcal{C}$-ultrahomogeneous if every isomorphism between induced subgraphs belonging to $\mathcal{C}$ extends to an automorphism of G. We could find no reference to any corresponding notion of $\mathcal{C}$-homogeneous graphs comparable with the $\mathcal{C}$-ultrahomogeneous graphs of [7], in the sense of Ronse's cited result [8]. In the present paper, the following notion of $\mathcal{C}$-homogeneity between isomorphic induced components in $\mathcal{C}$ anchored at arcs is considered.

Definition 1.1 *A graph G is $\vec{\mathcal{C}}$-homogeneous if, for any two isomorphic induced subgraphs $X_1, X_2 \in \mathcal{C}$ in G and any arcs v_1w_1 of X_1 and v_2w_2 of X_2, there exists an isomorphism $f : X_1 \to X_2$ extending to an automorphism of G and such that $f(v_1) = v_2$ and $f(w_1) = w_2$, that is $f(v_1w_1) = v_2w_2$.*

If, in Definition 1.1, $X_1 = X_2$, then X_1 (and hence any graph in $\mathcal{C}$ induced in G) must be 1-arc transitive [5]. If $\mathcal{C}$ is the minimal class containing two nonisomorphic graphs X_1 and X_2, then a $\vec{\mathcal{C}}$-homogeneous graph is said to be $\{\vec{X}_1, \vec{X}_2\}$*-homogeneous*. Clearly, any $\mathcal{C}$-ultrahomogeneous graph is $\vec{\mathcal{C}}$-homogeneous. In particular, the $\{K_4, K_{2,2,2}\}$-ultrahomogeneous graph G_3^1 of [3] is $\{\vec{K}_4, \vec{K}_{2,2,2}\}$-homogeneous.

Let $(r, \sigma) \in \mathbf{Z}^2$ with $r > 2$ and $\sigma \in (0, r-1)$. Let $t = 2^{\sigma+1} - 1$ and $s = 2^{r-\sigma-1}$. Let K_{2s} be the complete graph on $2s$ vertices and $T_{ts,t}$ be the t-partite (Turán) graph on s vertices per part (a total of ts vertices). Continuing the cited work of [3], a construction below make us conjecture that there exists a connected $\{\vec{T}_{ts,t}, \vec{K}_{2s}\}$-homogeneous graph G_r^σ that is not $\{\vec{T}_{ts,t}, \vec{K}_{2s}\}$-ultrahomogeneous for $r > 3$. In this paper, this conjecture is established for $r \leq 8$ and $\rho = r - \sigma \leq 5$.

The work of [7] dealt with the following 4 classes $\mathcal{C}$: **(A)** the complete graphs; **(B)** their complements (the empty graphs); **(C)** the disjoint unions of complete graphs; **(D)** their complements (the complete multipartite graphs).

Our present attempt seems to be the first one in the literature for a more heterogeneous class $\mathcal{C}'$ contained in (or even coinciding with) the union of the collections **(A)** and **(D)**, since K_{2s} is in **(A)** and $K_{ts,t}$ is in **(D)**. In fact, each proposed graph G_r^σ will coincide with some connected $\vec{\mathcal{C}}'$-homogeneous graph G expressible in a unique way, both as an edge-disjoint union U_1 of copies of $X_1 = K_{2s}$ and as an edge union U_2 of copies of $X_2 = T_{ts,t}$, and with:

(a) the class $\mathcal{C}'$ minimal for the property of containing any copies of X_1 and X_2;
(b) no more copies of X_i in G than in U_i, for $i=1,2$;
(c) no two copies of X_i in G sharing more than one vertex, for $i=1,2$;
(d) each edge of G shared by just one copy of X_1 and one of X_2, so that G is *edge-fastening.*

Clearly, such a G is regular. Moreover, the number $m_i(G,v)=m_i(G)$ of copies of X_i incident to each vertex v of G is independent of v, for $i=1,2$. This G will be said to be a *homogeneous* $\{\vec{X}_1\}_{\ell_1}^{m_1}\{\vec{X}_2\}_{\ell_2}^{m_2}$-*graph,* where ℓ_i = number of copies of X_i in G and $m_i=m_i(G)$, for $i=1,2$. We also say that G is $\{\vec{X}_1,\vec{X}_2\}$-*homogeneous.* If G is $\{X_1,X_2\}$-ultrahomogeneous, then G is said to be an *ultrahomogeneous* $\{X_1\}_{\ell_1}^{m_1}$ $\{X_2\}_{\ell_2}^{m_2}$-*graph.* It is not difficult to prove that the line graph of the n-cube is an ultrahomogeneous $\{K_n\}_{2^n}^{2}$ $\{K_{2,2}\}_{n(n-1)2^{n-3}}^{n-1}$-graph, for $3\le n\in\mathbf{Z}$. We could extend our definition above to say that the line graph of the 3-cube, the cuboctahedron, is an ultrahomogeneous $\{K_3\}_8^2\{C_4\}_6^2\{C_6\}_4^2$-graph.

If any of the graphs G_r^σ is an $\{\vec{X}_1\}_{\ell_1}^{m_1}\{\vec{X}_2\}_{\ell_2}^{m_2}$-graph, where $\min\{m_1,m_2\}>2$, then it is non-line-graphical: it cannot be a line graph of any other graph.

We pass to present the notions needed in order to define the graphs G_r^σ and establish their claimed properties.

2 Graphs of ordered pencils

We can identify the binary projective $(r-1)$-space $\mathbf{P}_2^{r-1}$ with the nonzero part of the field $\mathbf{F}_2^r$ of 2^r elements. This way, if $j\in[0,r-2]\cap\mathbf{Z}$, then each j-subspace of $\mathbf{P}_2^{r-1}$ equals the intersection of $\mathbf{F}_2^r\setminus\{\bar{0}\}$ with a corresponding $\mathbf{F}_2$-linear j-subspace of $\mathbf{F}_2^r$. In order to interpret this and keep up notation below, the empty set of $\mathbf{P}_2^{r-1}$ will be said to be a $(-j)$-*subspace* of $\mathbf{P}_2^{r-1}$, for every $0\le j\in\mathbf{Z}$.

Let $n=2^r-1\in\mathbf{Z}$. Each one of the n points $a_0a_1\dots a_{r-1}\neq\bar{0}$ in $\mathbf{P}_2^{r-1}$ is denoted by the integer it represents as a binary r-tuple with an hexadecimal read-out and the reading starting at the first nonzero a_i, $(i\in[0,r)\cap\mathbf{Z})$, up to a_{r-1}. This way, $\mathbf{Z}\cap(0,2^r)$ means $\mathbf{P}_2^{r-1}$ endowed with the natural ordering.

Now, the projective hyperplane $\mathbf{P}_2^{r-2}$ is identified with the $(r-2)$-subspace of $\mathbf{P}_2^{r-1}$ represented by $\mathbf{Z}\cap(0,2^{r-1})$ and called the *initial copy* of $\mathbf{P}_2^{r-2}$ in $\mathbf{P}_2^{r-1}$. Its points can be considered as the *directions of parallelism* of the affine space $A(r-1)$ obtained from $\mathbf{P}_2^{r-1}\setminus\mathbf{P}_2^{r-2}$ by puncturing the first entry, $(a_0=1)$, of its points, $a_0a_1\dots a_{r-1}$.

Each one of the $2^{r-2}-1$ $(r-3)$-subspaces S of the initial copy of $\mathbf{P}_2^{r-2}$ in $\mathbf{P}_2^{r-1}$ yields two $(r-2)$-subspaces of $\mathbf{P}_2^{r-1}$: **(A)** an $(r-2)$-subspace formed by the points of S and the *complements* in n of the points $i\in\mathbf{P}_2^{r-2}\setminus S$, namely the points $n-i$; **(B)** an $(r-2)$-subspace formed by the point $n=2^r-1$, the points i of S and their complements $n-i$ in n.

This representation of subspaces of $\mathbf{P}_2^{r-1}$, without parentheses and commas, is extensible to affine subspaces and their complementary subspaces in $\mathbf{P}_2^{r-1}$, considered as their *subspaces at infinity.* Similarly, any other subspace of $\mathbf{P}_2^{r-1}$ of dimension >0 is presentable via an initial copy of a lower-dimensional subspace.

Examples. $\mathbf{P}_2^2$ is formed by the nonzero binary 3-tuples 001, 010, 011, 100, 101, 110, 111, that we denote respectively by means of their hexadecimal integer forms: $1,2,3,4,5,6,7$. Now, $\mathbf{P}_2^2\subset\mathbf{P}_2^3$ is represented by $\{1,\dots,7\}$ immersed into $\{1,\dots,\ f=15\}$ by sending $1:=001$ onto $1:=0001$; $2:=010$ onto $2:=0010$, etc., that is: by prefixing a zero to each 3-tuple. Now, puncturing the first entry from the 4-tuples of $\mathbf{P}_2^3$ (and expressing the punctured entry between parentheses) yields: (0)001 as the direction of parallelism of the affine lines of $A(3)$ with point sets $\{(1)000,(1)001\}$, $\{(1)010,(1)011\}$, $\{(1)100,(1)101\}$, $\{(1)110,(1)111\}$; these affine lines are denoted $89,ab,cd,ef$, respectively. On the other hand, $\mathbf{P}_2^1$ is formed by the points 1,2,3 and the sole line 123. Also, $\mathbf{P}_2^1$ determines the planes $123ba98=123(f-4)(f-5)(f-6)(f-7)$ and $123fedc=123f(f-1)(f-2)(f-3)$ in $\mathbf{P}_2^3$, respectively.

Let A_0 be a σ-subspace of $\mathbf{P}_2^{r-1}$. The set of $(\sigma+1)$-subspaces of $\mathbf{P}_2^{r-1}$ that contain A_0 is the (r,σ)-*pencil* of $\mathbf{P}_2^{r-1}$ through A_0. A linearly ordered presentation of this is an (r,σ)-*ordered pencil* of $\mathbf{P}_2^{r-1}$ through A_0. Notice that there are $(2^{r-\sigma}-1)!$ (r,σ)-ordered pencils of $\mathbf{P}_2^{r-1}$ through A_0, since there are $2^{r-\sigma}-1$ $(\sigma+1)$-subspaces containing A_0 in $\mathbf{P}_2^{r-1}$. An (r,σ)-ordered pencil v of $\mathbf{P}_2^{r-1}$ through A_0 has the form $v=(A_0\cup A_1,\dots,A_0\cup A_{m_1})$, where $A_1,\dots,A_{m_1}$ are the nontrivial cosets of $\mathbf{F}_2^r$ mod its subspace $A_0\cup\{\bar{0}\}$, with $m_1 = 2^{r-\sigma} - 1$. A shorthand for this will be used throughout: we just write $v = (A_0, A_1, \ldots, A_{m_1})$ and consider $A_1, \ldots, A_{m_1}$ as the *non-initial* entries of v.

We define $\mathcal{G}_r^\sigma$ as the graph whose vertices are the (r,σ)-ordered pencils $v = (A_0, A_1, \ldots, A_{m_1}) = (A_0(v), A_1(v), \ldots, A_{m_1}(v))$ of $\mathbf{P}_2^{r-1}$, with an edge precisely between each two vertices $v = (A_0, A_1 \ldots, A_{m_1})$ and $v' = (A'_0, A'_1 \ldots, A'_{m_1})$ that satisfy:
1. $A_0 \cap A'_0$ is a $(\sigma-1)$-subspace of $\mathbf{P}_2^{r-1}$;
2. for each $1 \leq i \leq m_1$, $A_i \cap A'_i$ is a nontrivial coset of $\mathbf{F}_2^r$ mod $(A_0 \cap A'_0) \cup \{\bar{0}\}$;
3. $U(v,v') = \cup_{i=1}^{m_1}(A_i \cap A'_i)$ is an $(r-2)$-subspace of $\mathbf{P}_2^{r-1}$.
Here, item **3.** is needed only if $(r,\sigma) \neq (3,1)$, for it is implied by **1.-2.** if $(r,\sigma) = (3,1)$.

Examples. Let v_r^σ be the lexicographically smallest (r,σ)-ordered pencil in $\mathcal{G}_r^\sigma$ and let u_r^σ be its lexicographically smallest neighbor in $\mathcal{G}_r^\sigma$. For example:

$$\begin{array}{lll}
v_3^1{=}(1{,}23{,}45{,}67), & u_3^1{=}(2{,}13{,}46{,}57), & (U(v_3^1{,}u_3^1){=}347); \\
v_4^1{=}(1{,}23{,}45{,}67{,}89{,}ab{,}cd{,}ef), & u_4^1{=}(2{,}13{,}46{,}57{,}8a{,}9b{,}ce{,}df), & (U(v_4^1{,}u_4^1){=}3479bcf); \\
v_4^2{=}(123{,}4567{,}89ab{,}cdef), & u_4^2{=}(145{,}2367{,}89cd{,}abef, & (U_(v_4^2{,}u_4^2){=}16789ef).
\end{array}$$

We define G_r^σ to be the component of $\mathcal{G}_r^\sigma$ containing v_r^σ.
Remarks. **(a)** For each $(\sigma+1)$-subspace W of $\mathbf{P}_2^{r-1}$ and $i \in [1, m_1] \cap \mathbf{Z}$, there is a copy of $T_{ts,t}$ induced in $\mathcal{G}_r^\sigma$ by all the vertices $(A_0, A_1, \ldots, A_{m_1})$ of $\mathcal{G}_r^\sigma$ for which A_0 is a σ-subspace of W and $A_i = W \setminus A_0$. Let us denote this copy of $T_{ts,t}$ by $[(W)_i]_r^\sigma$. For example, the 4-vertex parts of the lexicographically first and last of the 7 copies of $T_{12,3}$ in G_4^1 incident to v_4^1, namely $[(\mathbf{P}_2^1)_1]_4^1 = [123_1]_4^1$ and $[1ef_7]_4^1$, are (columnwise):

$[123_1]_4^1$	$(1{,}23{,}45{,}67{,}89{,}ab{,}cd{,}ef)$	$(2{,}13{,}46{,}57{,}8a{,}9b{,}ce{,}df)$	$(3{,}12{,}47{,}56{,}8b{,}9a{,}cf{,}de)$
	$(1{,}23{,}45{,}67{,}ab{,}89{,}ef{,}cd)$	$(2{,}13{,}46{,}57{,}9b{,}8a{,}df{,}ce)$	$(3{,}12{,}47{,}56{,}9a{,}8b{,}de{,}cf)$
	$(1{,}23{,}67{,}45{,}89{,}ab{,}ef{,}cd)$	$(2{,}13{,}57{,}46{,}8a{,}9b{,}df{,}ce)$	$(3{,}12{,}56{,}47{,}8b{,}9a{,}de{,}cf)$
	$(1{,}23{,}67{,}45{,}ab{,}89{,}cd{,}ef)$	$(2{,}13{,}57{,}46{,}9b{,}8a{,}ce{,}df)$	$(3{,}12{,}56{,}47{,}9a{,}8b{,}cf{,}de)$
$\ldots$	$\ldots$	$\ldots$	$\ldots$
$[1ef_7]_4^1$	$(1{,}23{,}45{,}67{,}89{,}ab{,}cd{,}ef)$	$(e{,}2c{,}4a{,}68{,}79{,}5b{,}3d{,}1f)$	$(f{,}2d{,}4b{,}69{,}78{,}5a{,}3c{,}1e)$
	$(1{,}23{,}ab{,}89{,}67{,}45{,}cd{,}ef)$	$(e{,}2c{,}5b{,}79{,}68{,}4a{,}3d{,}1f)$	$(f{,}2d{,}5a{,}78{,}69{,}4b{,}3c{,}1e)$
	$(1{,}cd{,}45{,}89{,}67{,}ab{,}23{,}ef)$	$(e{,}3d{,}4a{,}79{,}68{,}5b{,}2c{,}1f)$	$(f{,}3c{,}4b{,}78{,}69{,}5a{,}2d{,}1e)$
	$(1{,}cd{,}ab{,}67{,}89{,}45{,}23{,}ef)$	$(e{,}3d{,}5b{,}68{,}79{,}4a{,}2c{,}1f)$	$(f{,}3c{,}5a{,}69{,}78{,}4b{,}2d{,}1e)$

The first and last of the 14 2-vertex parts of the 3 (columnwise) copies of $T_{14,7}$ in G_4^2 incident to v_4^2, namely $[(\mathbf{P}_2^2)_1]_4^2 = [1234567_1]_4^2$, $[12389ab_2]_4^2$ and $[123cdef_3]_4^2$, are:

$[1234567_1]_4^2$	$[12389ab_2]_4^2$	$[123cdef_3]_4^2$
$(123{,}4567{,}89ab{,}cdef)$	$(123{,}4567{,}89ab{,}cdef)$	$(123{,}4567{,}89ab{,}cdef)$
$(123{,}4567{,}cdef{,}89ab)$	$(123{,}cdef{,}89ab{,}4567)$	$(123{,}89ab{,}4567{,}cdef)$
$\ldots$	$\ldots$	$\ldots$
$(356{,}1247{,}8bde{,}9acf)$	$(39a{,}47de{,}128b{,}56cf)$	$(3de{,}479a{,}568b{,}12cf)$
$(356{,}1247{,}9acf{,}8bde)$	$(39a{,}56cf{,}128b{,}47de)$	$(3de{,}568b{,}479a{,}12cf)$

(b) For each $(r-1,\sigma-1)$-ordered pencil $U = (U_0, U_1, \ldots, U_{m_1})$ of an $(r-2)$-subspace of $\mathbf{P}_2^{r-1}$, there is a copy of K_{2s} in G_r^σ (where $U_0 = \emptyset$ in case $\sigma = 1$) induced by the vertices $(A_0, A_1, \ldots, A_{m_1})$ of G_r^σ having $A_i \supset U_i$, for $1 \leq i \leq m_1$. Let us denote this copy of K_{2s} by $[U]_r^\sigma = [U_0, U_1, \ldots, U_{m_1}]_r^\sigma$. For example, the induced copies of K_8 in G_4^1 incident to v_4^1 are:

$$\begin{array}{llll}
[\emptyset{,}2{,}4{,}6{,}8{,}a{,}c{,}e]_4^1 & {=}[2468ace]_4^1, & [\emptyset{,}3{,}4{,}7{,}8{,}b{,}c{,}f]_4^1 & {=}[3478bcf]_4^1, \\
[\emptyset{,}2{,}4{,}6{,}9{,}b{,}d{,}f]_4^1 & {=}[2469bdf]_4^1, & [\emptyset{,}3{,}4{,}7{,}9{,}a{,}d{,}e]_4^1 & {=}[3479ade]_4^1, \\
[\emptyset{,}2{,}5{,}7{,}8{,}a{,}d{,}f]_4^1 & {=}[2578adf]_4^1, & [\emptyset{,}3{,}5{,}6{,}8{,}b{,}d{,}e]_4^1 & {=}[3568bde]_4^1, \\
[\emptyset{,}2{,}5{,}7{,}9{,}b{,}c{,}e]_4^1 & {=}[2579bce]_4^1, & [\emptyset{,}3{,}5{,}6{,}9{,}a{,}c{,}f]_4^1 & {=}[3569acf]_4^1,
\end{array}$$

where we may use, for $\sigma = 1$, the shorter notation shown to the right, without the symbol $\emptyset$ and the commas. The induced copies of K_4 in G_4^2 are:

$$\begin{array}{llllll}
[1{,}45{,}89{,}cd]_4^2, & [1{,}67{,}89{,}ef]_4^2, & [2{,}57{,}8a{,}df]_4^2, & [2{,}46{,}8a{,}ce]_4^2, & [3{,}47{,}8b{,}cf]_4^2, & [3{,}56{,}8b{,}de]_4^2, \\
[1{,}45{,}ab{,}ef]_4^2, & [1{,}67{,}ab{,}cd]_4^2, & [2{,}57{,}9b{,}ce]_4^2, & [2{,}46{,}9b{,}df]_4^2, & [3{,}47{,}9a{,}de]_4^2, & [3{,}56{,}9a{,}cf]_4^2.
\end{array}$$

The vertices of each such copy $[U]_r^\sigma$ of K_{2s} can be displayed as a product of arrays of subsets of the form $\{A_{i,j}\} = \{X_{i,j}\} \times \{Y_{i,j}\} = \{X_{i,j} \cup Y_{i,j}\}$. For example, $[U]_r^\sigma = [\emptyset, 2, 4, 6, 8, a, c, e]_4^1$ can be displayed as:

$\{X_{i,j} \cup Y_{i,j}\} =$

$$\begin{array}{c}
(1,23,45,67,89,ab,cd,ef)\\
(3,12,47,56,8b,9a,cf,de)\\
(5,27,14,36,8d,af,9c,be)\\
(7,25,34,16,8f,ad,bc,9e)\\
(9,2b,4d,6f,18,3a,5c,7e)\\
(b,29,4f,6d,38,1a,7c,5e)\\
(d,2f,49,6b,58,7a,1c,3e)\\
(f,2d,4b,69,78,5a,3c,1e)
\end{array}
=
\begin{array}{c}
(\emptyset,2,4,6,8,a,c,e)\\
(\emptyset,2,4,6,8,a,c,e)\\
(\emptyset,2,4,6,8,a,c,e)\\
(\emptyset,2,4,6,8,a,c,e)\\
(\emptyset,2,4,6,8,a,c,e)\\
(\emptyset,2,4,6,8,a,c,e)\\
(\emptyset,2,4,6,8,a,c,e)\\
(\emptyset,2,4,6,8,a,c,e)
\end{array}
\times
\begin{array}{c}
(1,3,5,7,9,b,d,f)\\
(3,1,7,5,b,9,f,d)\\
(5,7,1,3,d,f,9,b)\\
(7,5,3,1,f,d,b,9)\\
(9,b,d,f,1,3,5,7)\\
(b,9,f,d,3,1,7,5)\\
(d,f,9,b,5,7,1,3)\\
(f,e,b,9,7,5,3,1)
\end{array}
= \{X_{i,j}\} \times \{Y_{i,j}\}$$

where $\{X_{i,j}\}$ has constant columns and each one of its rows as $U = (\emptyset, 2, 4, 6, 8,\ a, c, e)$, and where $Y_{i,j} = A_j \setminus U_j$. The sets $Y_{i,j}$, for each fixed $i \in [2, m_1] \cap \mathbf{Z}$, form a permutation of the top sets $Y_{1,j}$. With the numbers $j \in [0, m_1] \cap \mathbf{Z}$ representing the top sets $Y_{1,j}$, the sets $Y_{i,j}$ are seen to form a Latin square via induction steps on $\rho = r - \sigma$, indicated by arrows here:

$$J \rightarrow \left(\begin{array}{ll} J & J + 2^{\rho} \\ J + 2^{\rho} & J \end{array} \right) : \
\begin{array}{c}
(1,2)\\
(2,1)
\end{array}
\ \rightarrow \
\begin{array}{c}
(1,2,3,4)\\
(2,1,4,3)\\
(3,4,1,2)\\
(4,3,2,1)
\end{array}
\ \rightarrow \
\begin{array}{c}
(1,2,3,4,5,6,7,8)\\
(2,1,4,3,6,5,8,7)\\
(3,4,1,2,7,8,5,6)\\
(4,3,2,1,8,7,5,4)\\
(4,5,7,8,1,2,3,4)\\
(6,5,8,7,2,1,4,3)\\
(7,8,5,6,3,4,1,2)\\
(8,7,6,5,4,3,2,1)
\end{array}$$

where J is the $2^{\rho} \times 2^{\rho}$-square matrix these j's form. For example, the copy $[1, 45, 89, cd]_4^2$ of K_4 in G_4^2 is expressible as follows:

$$\begin{array}{c}
(123,4567,89ab,cdef)\\
(167,2345,89ef,abcd)\\
(1ab,45ef,6789,23cd)\\
(1ef,45ab,2389,67cd)
\end{array}
=
\begin{array}{c}
(1,45,89,cd)\\
(1,45,89,cd)\\
(1,45,89,cd)\\
(1,45,89,cd)
\end{array}
\times
\begin{array}{c}
(23,67,ab,ef)\\
(67,23,ef,ab)\\
(ab,ef,23,67)\\
(ef,ab,67,23)
\end{array}$$

(c) We want to prove that G_r^{σ} is a homogeneous $\{\vec{T}_{ts,t}\}_{\ell_1}^{m_1}\{\vec{K}_{2s}\}_{\ell_2}^{m_2}$-graph with $m_1 = 2^{\rho} - 1 = 2^{r-\sigma} - 1, m_2 = 2s(2^{\sigma} - 1), \ell_1 = \frac{m_1}{st}|V(G_r^{\sigma})|, \ell_2 = (2^{\sigma} - 1)|V(G_r^{\sigma})|$ and $|V(G_r^{\sigma})| = \binom{r}{\sigma}_2 \Pi_{i=1}^{\rho}(2^{i-1}(2^i - 1)) = \Pi_{i=1}^{\rho}(2^{i-1}(2^{i+\sigma} - 1)) = O(2^{(r-1)^2})$, where the Gaussian binomial coefficient $\binom{r}{\sigma}_2 = \Pi_{i=1}^{\rho} \frac{2^{i+\sigma}-1}{2^i-1}$ is the number of different σ-subspaces A_0 of $\mathbf{P}_2^{r-1}$. We start by establishing some properties of $\mathcal{G}_r^{\sigma}$. The initial case is the main result of [3], stating that the graph $\mathcal{G}_3^1 = G_3^1$ is a connected ultrahomogeneous 12-regular $\{K_4\}_{42}^4\ \{K_{2,2,2}\}_{21}^3$-graph of order 42 and diameter 3. As expressed in Section 1, we prove our claims for small values of r and $\rho = r - \sigma$, ($r \leq 9$, $\rho \leq 5$). The general case would follow from Conjecture 5.2.

Theorem 2.1 *$\mathcal{G}_r^{\sigma}$ has order $\binom{r}{\sigma}_2 m_1!$ and regular degree $m_1 s(t-1)$. Moreover, $\mathcal{G}_r^{\sigma}$ is uniquely representable as an edge-disjoint union of $m_1|V(\mathcal{G}_r^{\sigma})|s^{-1}t^{-1}$ (resp. $(2^{\sigma} - 1)|V(\mathcal{G}_r^{\sigma})|$) copies of $T_{ts,t}$ (resp. K_{2s}) and has exactly m_1 (resp. m_2) copies of $T_{ts,t}$ (resp. K_{2s}) incident at each vertex, with no two such copies sharing more than one vertex and each edge of $\mathcal{G}_r^{\sigma}$ in exactly one copy of $T_{ts,t}$ (resp. K_{2s}).*

Proof. The number of σ-subspaces F' in $\mathbf{P}_2^{r-1}$ is $\#F' = \binom{r}{\sigma}_2$. For each such F' taken as initial entry A_0 of some vertex v of $\mathcal{G}_r^{\sigma}$, there are m_1 classes mod $F' \cup \overline{0}$ permuted and distributed from left to right into the remaining positions A_i of v. Thus, $|\mathcal{G}_r^{\sigma}| = (\#F')m_1!$

Each v of $\mathcal{G}_r^{\sigma}$ is the sole intersection vertex of exactly m_1 copies of $T_{ts,t}$. Since the degree of $T_{ts,t}$ is $s(t-1)$, then the degree of $\mathcal{G}_r^{\sigma}$ is $m_1 s(t-1)$. The edge numbers of $T_{ts,t}$ and $\mathcal{G}_r^{\sigma}$ are respectively $s^2t(t-1)/2$ and $m_1 s(t-1)|V(\mathcal{G}_r^{\sigma})|/2$, so that $\mathcal{G}_r^{\sigma}$ is the edge-disjoint union of $m_1|V(\mathcal{G}_r^{\sigma})|s^{-1}t^{-1}$ copies of $T_{ts,t}$. No other copies of $T_{ts,t}$ exist in $\mathcal{G}_r^{\sigma}$. □

3 On the automorphism group $\mathcal{A}(G_r^{\sigma})$ of G_r^{σ}

The neighbors of v_r^{σ} in G_r^{σ} induce a subgraph $N_{G_r^{\sigma}}(v_r^{\sigma})$ of G_r^{σ}. Consider the vertex set $W_r^{\sigma} = \{w \in V(G_r^{\sigma}) : A_0(w) = A_0(v_r^{\sigma})\}$. For every $w \in W_r^{\sigma}$, consider the automorphism $h_w \in \mathcal{A}(G_r^{\sigma})$ that takes v_r^{σ} onto w, given by the permutation of the non-initial entries of w with respect to those of v_r^{σ}. To characterize $\mathcal{A}(G_r^{\sigma})$, it suffices to determine its subgroup $\mathcal{N}_r^{\sigma} = \mathcal{A}(N_{G_r^{\sigma}}(v_r^{\sigma}))$ and the automorphisms h_w. This is treated below from Subsection 4.1 on, where $\{h_w : w \in W_r^{\sigma})\}$ is endowed with the structure of a group $\mathcal{H}_{\rho}$. But

$|\mathcal{A}(G_r^\sigma)| = |\mathcal{N}_r^\sigma|.|\mathcal{H}_\rho|$ and each $w \in V(G_r^\sigma)$ will have $\mathcal{A}(N_{G_r^\sigma}(w)) = h_w^{-1}\mathcal{N}_r^\sigma h_w$, (composition from left to right), so $\mathcal{A}(G_r^\sigma)$ will be a semidirect product $\mathcal{N}_r^\sigma \times_\lambda \mathcal{H}_\rho$ with $\lambda : \mathcal{H}_\rho \to \mathcal{A}(\mathcal{N}_r^\sigma)$ given by $\lambda(h_w)$ sending $k \in \mathcal{N}_r^\sigma$ onto $h_w^{-1}kh_w$, that is to say: $\lambda(h_w) = (k \mapsto h_w^{-1}kh_w : k \in \mathcal{N}_r^\sigma)$.

A set of generators for $\mathcal{N}_r^\sigma$ will be given by means of products of transpositions of the form (α, β), where α, β are affine σ-subspaces of $\mathbf{P}_2^{r-1}$, each α and β having a common $(\sigma-1)$-subspace $\theta_{\alpha,\beta}$ at infinity. We define the *affine difference* $\chi_{\alpha,\beta}$ of such a pair α, β as the affine σ-subspace of $\mathbf{P}_2^{r-1}$ composed by the third points in the lines determined by each pair of points $b \in \alpha$ and $c \in \beta$. Here, it suffices to take such third points for some fixed $b \in \alpha$ and variable $c \in \beta$. Each transposition (α, β) will be denoted $[\theta_{\alpha,\beta}.\chi_{\alpha,\beta}(\alpha,\beta)]$, stressing the subspace at infinity and the affine difference associated to (α, β). Let $0 < h \in \mathbf{Z}$. A product of transpositions (α_i, β_i), for $1 \leq i \leq h$, with a common $\theta_{\alpha_i,\beta_i} = \theta$ and a common $\chi_{\alpha_i,\beta_i} = \chi$ is indicated $[\theta.\chi \prod_{i=1}^h (\alpha_i, \beta_i)]$. This could simply be written $\prod_{i=1}^h (\alpha_i, \beta_i)$, that is as a permutation of the affine σ-subspaces of $\mathbf{P}_2^{r-1}$ (with no reference to lines at infinity or affine differences). The points of $\mathbf{P}_2^{r-1}$ which form part of θ and χ but are outside those pairs of parentheses, namely $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots, (\alpha_h, \beta_h)$, are clearly fixed points of the resulting permutation ϕ. A permutation ψ of affine σ-subspaces of $\mathbf{P}_2^{r-1}$ is obtained from such a ϕ by replacing each of the numbers in its entries by the floor of its $(\frac{1}{2^\sigma})$-th part, and if some of the resulting transpositions coincide, setting only one of these repeated transpositions in ψ. Then, the composition $\tau = \phi.\psi = \psi.\phi$ is an automorphism of G_r^σ that we express by writing $\phi = \phi^\tau$ and $\psi = \psi^\tau$, namely $\tau = \phi^\tau.\psi^\tau = \psi^\tau.\phi^\tau$. In this context, an empty pair of parentheses, e.g. (), stands for the identity permutation. It can be seen that at least for $r \leq 8$, a set of generators $\tau = \phi^\tau.\psi^\tau$ of $\mathcal{A}(G_r^\sigma)$ is formed by those τ's in the following items:

(A) Given a point $\pi \in \mathbf{P}_2^\rho = \mathbf{P}_2^{r-\sigma}$ and an $(r-2)$-subspace α of $\mathbf{P}_2^{r-1}$ containing $\{\pi\} \cup \mathbf{P}_2^{\sigma-1}$, let $\phi^\tau = g(\pi, \alpha)$ be the product of the transpositions of pairs of affine $(\sigma-1)$-spaces of $\mathbf{P}_2^{r-1}$ with a common $(\sigma-1)$-subspace at infinity in $\mathbf{P}_2^{\sigma-1}$ and a corresponding common affine difference containing π and contained in $\alpha \backslash \mathbf{P}_2^{\sigma-1}$. Let $g'(\pi, \alpha)$ be the ψ^τ associated to ϕ^τ. Some examples of triples $(\tau = g(\pi, \alpha).g'(\pi, \alpha), \pi, \alpha)$ are, (in hexadecimal notation or its continuation in the English alphabet from $10 = a$ to $15 = f$ up to $31 = v$), as follows:

G_3^1:
$([\emptyset.2(4,6)(5,7)].1(2,3), \quad \pi=2, \quad \alpha=123);$
$([\emptyset.3(4,7)(5,6)].1(2,3), \quad \pi=3, \quad \alpha=123);$
$([\emptyset.6(2,4)(3,5)].3(1,2), \quad \pi=6, \quad \alpha=167);$
$([\emptyset.1(2,3)(6,7)].(), \quad \pi=1, \quad \alpha=145).$

G_4^1:
$([\emptyset.2(8,a)(9,b)(c,e)(d,f)].1(4,5)(6,7), \quad \pi=2, \quad \alpha=1234567);$
$([\emptyset.4(8,c)(9,d)(a,e)(b,f)].2(4,6)(5,7), \quad \pi=4, \quad \alpha=1234567);$
$([\emptyset.2(4,6)(5,7)(8,a)(9,b)].1(2,3)(4,5), \quad \pi=2, \quad \alpha=123cdef);$
$([\emptyset.c(4,8)(5,9)(6,a)(7,b)].6(2,4)(3,5), \quad \pi=c, \quad \alpha=123cdef);$
$([\emptyset.5(8,d)(9,c)(a,f)(b,e)].2(4,6)(5,7); \quad \pi=5, \quad \alpha=1234567);$
$([\emptyset.1(2,3)(6,7)(a,b)(e,f)].(); \quad \pi=1, \quad \alpha=14589cd);$
$([\emptyset.6(2,4)(3,5)(a,c)(b,d)].3(1,2)(5,6); \quad \pi=6, \quad \alpha=16789ef).$

G_4^2:
$([1.45(89,cd)(ab,ef)][2.46(8a,ce)(9b,df)][3.47(8b,cf)(9a,de)].1(2,3), \quad \pi=4, \quad \alpha=1234567);$
$([1.cd(45,89)(67,ab)][2.ce(46,8a)(57,9b)][3.cf(47,8b)(56,9a)].3(1,2), \quad \pi=c, \quad \alpha=123cdef);$

G_5^2:
$([1.op(89,gh)(ab,ij)(cd,kl)(ef,mn)][2.oq(8a,gi)(9b,hj)(ce,km)(df,ln)]$
$[3.or(8b,gj)(9a,hi)(cf,kn)(de,lm)].6(2,4)(3,5), \quad \pi=o, \quad \alpha=1234567opqrstuv);$

G_5^3
$([123.89ab(ghij,opqr)(klmn,stuv)][145.89cd(ghkl,opst)(ijmn,qruv)]$
$[167.89ef(ghmn,opuv)(ijkl,qrst)][246.8ace(gikm,oqsu)(hjln,prtv)]$
$[257.8adf(giln,oqtv)(hjkm,prsu)][347.8bcf(gjkn,orsv)(hilm,pqtu)]$
$[356.8bde(gjkn,ortu)(hilm,pqsv)].1(2,3), \quad \pi=8, \quad \alpha=123456789abcdef).$

(B) Given a $(\sigma-1)$-subspace π in $\mathbf{P}_2^{\sigma-1}$ and an $(r-2)$-subspace α of $\mathbf{P}_2^{r-1}$ containing $\mathbf{P}_2^{\rho-1}$, let $\phi^\tau = h(\pi, \alpha)$ be the product of the transpositions of the pairs of affine σ-subspaces of $\mathbf{P}_2^{r-1}$ not contained in $\alpha \setminus \mathbf{P}_2^{\sigma-1}$, with $(\sigma-1)$-subspace at infinity π and corresponding affine difference $\mathbf{P}_2^{\sigma-1} \setminus \pi$. In each case, $\psi^\tau = ()$. Some examples of (τ, π, α) here are:

G_4^2:
$([1.23(89,ab)(cd,ef)].(), \quad \pi=1, \quad \alpha=1234567);$
$([1.23(45,67)(cd,ef)].(), \quad \pi=1, \quad \alpha=12389ab);$
$([1.23(45,67)(89,ab)].(), \quad \pi=1, \quad \alpha=123cdef);$
$([2.13(8a,9b)(ce,df)].(), \quad \pi=2, \quad \alpha=1234567);$
$([2.13(46,57)(ce,df)].(), \quad \pi=2, \quad \alpha=12389ab);$
$([2.13(46,57)(8a,9b)].(), \quad \pi=2, \quad \alpha=123cdef);$
$([3.12(8b,9a)(cf,de)].(), \quad \pi=3, \quad \alpha=1234567);$
$([3.12(47,56)(cf,de)].(), \quad \pi=3, \quad \alpha=12389ab);$
$([3.12(47,56)(8b,9a)].(), \quad \pi=3, \quad \alpha=123cdef);$

G_5^2:
$([2.13(gi,hj)(km,ln)(oq,pr)(su,tv)].(), \quad \pi=2, \quad \alpha=123456789abcdef);$
$([2.13(8a,9b)(ce,df)(pr,oq)(su,tv)].(), \quad \pi=2, \quad \alpha=1234567ghijklmn);$
$([1.23(89,ab)(cd,ef)(op,qr)(st,uv)].(), \quad \pi=1, \quad \alpha=1234567ghijklmn);$
$([1.23(gh,ij)(kl,mn)(op,qr)(st,uv)].(), \quad \pi=1, \quad \alpha=123456789abcdef);$
$([3.12(8b,9a)(cf,de)(or,pq)(sv,tu)].(), \quad \pi=3, \quad \alpha=1234567ghijklmn);$
$([3.12(gj.hi)(kn,ln)(or,pq)(st,tu)].(), \quad \pi=3, \quad \alpha=123456789abcdef);$

G_5^3:
$[(347.1256(gjkn,hilm)(pqsv,ortu)].(), \quad \pi=347, \quad \alpha=123456789abcdef).$

(C) Given a $(\sigma-1)$-subspace π of $\mathbf{P}_2^{\sigma-1}$ and an $(r-2)$-subspace α of $\mathbf{P}_2^{r-1}$ with $\pi \in \alpha \cap \mathbf{P}_2^{\sigma-1}$, let ϕ^τ be the product of the transpositions of pairs of affine σ-subspaces of $\mathbf{P}_2^{r-1}$ not contained in α, with $(\sigma-1)$-subspace at infinity contained in α and affine difference contained in α and containing π. Again, $\psi^\tau = ()$. Some examples of (τ, π, α) here are:

G^2_4: $([4.15(26,37)][5.14(27,36)][8.19(2a,3b)][9.18(2b,3a)][c.1d(2e,3f)][d.1c(2f,3e)].()$, $\pi=1$, $\alpha=1459cd)$;
$([4.37(15,26)][7.34(16,25)][8.3b(19,2a)][b.38(1a,29)][c.3f(1d,2e)][f.3c(1e,2d)].()$, $\pi=3$, $\alpha=3478bcf)$;

G^2_5: $([4.37(15,26)][7.34(16,25)][8.3b(19,2a)][b.38(1a,29)][c.3f(1d,2e)][f.3c(1e,2d)]$
$[g.3j(1h,2i)][j.3g(1i,2h)][k.3n(1l,2m)][n.3k(1m,2l)][o.3r(1p,2q)][r.3o(1a,2p)]$
$[v.3s(1u,2t)][s.3v(1t,2u)].()$, $\pi=3$, $\alpha=3478bcfgjknorsv)$;

G^3_5: $([189.67ef(23ab,45cd)][1ef.6789(23cd,45ab)][1gh.67mn(23ij,45kl)]$
$[1mn.67gh(23kl,45ij)][1op.67uv(23qr,45st)][1uv.67op(23st,45qr)].()$, $\pi=167$, $\alpha=16789efghmnopuv)$;

$([189.23ab(45cd,67ef)][1ab.2389(45ef,67cd)]1kl.23mn(45gh,67ij)]$
$[1mn.23kl(45ij,67gh)][1st.23uv(45op,67qr)][1uv.23st(45qr,67op)].()$, $\pi=123$, $\alpha=12389abklmnstuv)$.

From the generators of $\mathcal{N}^\sigma_r$ presented in items (A)-(C) above, the following was established for $r \leq 8$ via the software Magma and conjectured in general for $r > 8$.

Theorem 3.1 *For $\sigma > 0$ and $\rho = r - \sigma > 1$,* (so that $r > 2$), let

$$\begin{array}{l} A = 2^{\sigma+1} - 1 + (\rho - 2)(2^\sigma + 1) + \max(\rho - 3, 0), \\ B = \Pi^\rho_{i=1}(2^i - 1) \text{ and} \\ C = (2^\sigma - 1)! \end{array}$$

Then, at least for $r \leq 8$, the cardinality of $\mathcal{N}^\sigma_r$ is 2^ABC, where the last term in the sum expressing A differs from $\rho - 3$ only if $\rho = 2$. ☐

4 On order and diameter of G^σ_r

In order to establish the properties of G^σ_r claimed in Remark (c) of Section 2, we need to estimate its order and diameter, for each $(r, \sigma) \in \mathbf{Z}^2$ with $r > 3$ and $\sigma \in (0, r - 1)$. The diameter of G^σ_r is realized by the distance from $v^\sigma_r = (A_0(v^\sigma_r), A_1(v^\sigma_r), \ldots, A_{m_1}(v^\sigma_r))$ to some vertex $w \in W^\sigma_r \setminus \{v^\sigma_r\}$, with W^σ_r as in Section 3, above.

4.1 Auxiliary graph H_ρ

In the square graph $(G^\sigma_r)^2$, consider the graph H induced by W^σ_r. Clearly, $v^\sigma_r \in V(H)$. Moreover, H depends only on $\rho = r - \sigma$, so we write $H = H_\rho$. Furthermore,

$$Diameter(G^\sigma_r) \leq 2 \times Diameter(H_\rho).$$

Consider the case $(r, \sigma) = (3, 1)$. We write $B_1 = 23, B_2 = 45, B_3 = 67$, independently of the entries that these pairs may occupy in a vertex of H_2 $(= K_{3,3})$. We assign to each vertex v of H_2 the permutation that maps the subindices i of the entries A_i of v, $(i = 1, 2, 3)$, into the subindices j of the pairs B_j correspondingly filling those entries A_i. This yields the following bijection from $V(H_2) = W^1_3$ onto the group $K = S_3$ of permutations of the point set of the projective line $\mathbf{P}^1_2$, (with permutation expressed in cycle notation):

(1,23,45,67)	→	123	(1,23,67,45)	→	1(23)
(1,45,23,67)	→	3(12)	(1,45,67,23)	→	(123)
(1,67,23,45)	→	(132)	(1,67,45,23)	→	2(13)

where each permutation on the right side of '→' is presented with its nontrivial cycles written, as usual, between parentheses and with the fixed points expressed in front and out of any pair of parentheses, for convenience of reference.

More generally, there is a bijection from $V(H_\rho)$ onto a group $\mathcal{H}_\rho$ of permutations of the point set of $\mathbf{P}^{\rho-1}_2$. The elements of $\mathcal{H}_\rho$, that we will call *$\mathcal{A}$-permutations*, form an auxiliary notation for the vertices of H_ρ. Thus, we denote $V(H_\rho) = \mathcal{H}_\rho$. For example, $v^\sigma_r \in V(H_\rho)$ is now invested as the identity permutation $I_\rho = 123\ldots2^\rho$, with fixed-point set $\mathbf{P}^{\rho-1}_2 = 123\ldots2^\rho$. Observe that $\mathcal{H}_\rho$ is formed by permutations of the non-initial entries of ordered pencils that are vertices of G_r^σ, as were the permutations ψ^τ in Section 3, but in each of the present cases the corresponding ϕ^τ composing with ψ^τ an automorphism τ of G_r^σ is the identity $()$, since this τ takes v_r^σ into some vertex $w \in W_r^\sigma$.

An ascending sequence $V(H_2) \subset V(H_3) \subset \ldots \subset V(H_\rho) \subset \ldots$ of $\mathcal{A}$-permutation groups is generated via the embeddings $\Psi_\rho : V(H_{\rho-1}) \to V(H_\rho)$, $(\rho > 2)$, defined by $\Psi_\rho(\phi)$ equal to the product of the $\mathcal{A}$-permutation ϕ of $\mathbf{P}_2^{\rho-2} \subset \mathbf{P}_2^{\rho-1}$ times the permutation obtained from ϕ by replacing each of its symbols i by the new symbol $m_1 - i$, with m_1 becoming a fixed point of $\Psi_\rho(\phi)$. Let us call this construction of $\Psi_\rho(\phi)$ out of ϕ the *doubling* of ϕ. For example, $\Psi_3 : V(H_2) \to V(H_3))$ maps the elements of $V(H_2)$ as follows:

$$\begin{array}{lll|lll}
123 & \to & 7123654 \quad =1234567 & 1(23) & \to & 71(23)6(54)=167(23)(45) \\
(123) & \to & 7(123)(654)=7(123)(465) & 3(12) & \to & 73(12)4(65)=347(12)(56) \\
(132) & \to & 7(132)(645)=7(132)(456) & 2(13) & \to & 72(13)5(64)=257(13)(46)
\end{array}$$

where each resulting $\mathcal{A}$-permutation in $V(H_3)$ is rewritten to the right, by expressing, from left to right and lexicographically, first the fixed points and then the cycles.

The 3 $\mathcal{A}$-permutations in $V(H_3)$ listed rightmost in the exemplified assignment above are of the form $abc(de)(fg)$, where ade and afg are lines of $\mathbf{P}_2^3$, namely: 123 and 145, for 167(23)(45); 312 and 356, for 347(12)(56); 213 and 246, for 257(13)(46).

A point of $\mathbf{P}_2^{\rho-1}$ playing the role of a in a product Π of $2^{\rho-2}$ disjoint transpositions, as in the just cited rightmost $\mathcal{A}$-permutations, is called the *pivot* of Π. For each point a of $\mathbf{P}_2^2$, there are 3 $\mathcal{A}$-permutations in $V(H_3)$ having a as its pivot. For example, the $\mathcal{A}$-permutations in $V(H_3)$ having pivot 1 are: 123(45)(67), 145(23)(67) and 167(23)(45).

For each $(\rho - 2)$-subspace Q of $\mathbf{P}_2^{\rho-1}$ and each point $a \in Q$, a (Q,a)*-transposition* is defined as a permutation (bc) such that there is a line $abc \subset \mathbf{P}_2^{\rho-1}$ with $bc \cap Q = \emptyset$. For each pair (Q,a) formed by a permutation Q and a point a as above, there are exactly $2^{\rho-2}$ (Q,a)-transpositions. Their product is an $\mathcal{A}$-permutation in $V(H_\rho)$ that we call the (Q,a)*-permutation*, $p(Q,a)$, with Q as its fixed-point set and a as its pivot.

The (Q,a)-permutations $p(Q,a)$ in $V(H_\rho)$ act as a set of generators for the group $V(H_\rho)$. In fact, all elements of $V(H_\rho)$ can be obtained from the (Q,a)-permutations by means of reiterated multiplications.

4.2 A vertex J_ρ of H_ρ at maximum distance from I_ρ

For $\rho > 1$, a particular element $J_\rho \in V(H_\rho) \setminus V(H_{\rho-1}^1)$ at maximum distance from I_ρ, (see Theorem 4.1 below), is obtained as a product $I_\rho = p_\rho q_\rho$ with:
(A) $p_\rho = p(Q, 2^{\rho-1})$, where Q is the $(\rho-2)$-subspace of $\mathbf{P}_2^{\rho-1}$ containing $2^{\rho-1}$ as well as all of $\mathbf{P}_2^{\rho-3}$. For example

$$\begin{array}{l}
p_2 = 2(13), \quad p_3 = 415(26)(37), \quad p_4 = 81239ab(4c)(5d)(6e)(7f), \\
p_5 = g1234567hijklmn(8o)(9p)(aq)(br)(cs)(dt)(eu)(fv), \quad p_6 = \ldots,
\end{array}$$

where hexadecimal notation, or its continuation in the English alphabet, is used.
(B) q_ρ defined inductively by $q_2 = 3(12)$ and $q_{\rho+1} = \Psi_\rho(p_\rho q_\rho)$, for $\rho > 1$, where Ψ_ρ is as in Subsection 4.1.

Initial examples of J_ρ, with products indicated by dots '$\cdot$', are

$$\begin{array}{l}
J_2{=}2(13){\cdot}3(12){=}(132); \\
J_3{=}415(26)(37){\cdot}7(132)(645){=}(1372456); \\
J_4{=}81239ab(4c)(5d)(6e)(7f){\cdot}f(1372456)(ec8dba9){=}(137f248d6c5ba9e); \\
J_5{=}g1234567hijklmn(8o)(9p)(aq)(br)(cs)(dt)(eu)(fv){\cdot} \\
\qquad\qquad (137f248d6c5ba9e)(usogtrnipjqklmh){=} \\
\quad {=}(137fv248gt6codraklmhu)(5bnipes)(9jq).
\end{array}$$

4.3 Types of vertices of H_ρ and type-distance relation

A way of expressing any permutation $v = J_2, J_3, J_4, J_5, \ldots$ in Subsection 4.2 is by means of the accompaniment of an underlying permutation u:

$$\begin{array}{l|l|l|l}
v{=}(132) & v{=}(1372456) & v{=}(137f248d6c5ba9e) & v{=}(137fv248gt6codraklmhu)(5bnipes)(9jq) \\
u{=}(213) & u{=}(2456137) & u{=}(248d6c5ba9e137f) & u{=}(248gt6codraklmhu137fv)(es5bnip)(q9j)
\end{array}$$

where each symbol b_i in a cycle of u and located under a symbol a_i of a cycle $(a_0a_1\dots a_{x-1})$ of v determines a line $a_ib_ia_{i+1}$ of $\mathbf{P}_2^{\rho-1}$, (with $i+1$ taken mod x). Each $\mathcal{A}$-permutation v, like for example any of J_2, J_3, J_4, J_5, will likewise be written with the accompaniment of a second similar expression u along a level underlying that of v. In this context, we say that:
(a) b_i is a *difference symbol*, (or *ds*), of v; **(b)** $(b_0b_1\dots b_{x-1})$ is the *ds-companion cycle* of $(a_0a_1\dots a_{x-1})$; and **(c)** u is the *ds-level* of v.

Each cycle $(a_0a_1\dots a_{x-1})$ of J_2, J_3, J_4, J_5 was expressed by means of a pair of cycles, $(a_0a_1\dots a_{x-1})$ and $(b_0b_1\dots b_{x-1})$, one on top of the other, in v and u respectively. Notice that these two cycles differ by just a shift of $(b_0b_1\dots b_{x-1})$ with respect to $(a_0a_1\dots a_{x-1})$ in the amount of, say, y positions. The values of y are, for our 4 examples: $J_2 : y = 1$; $J_3 : y = 4$; $J_4 : y = 12$; $J_5 : y = 16, 3, 1$, (one for each cycle of J_5).

For $\rho > 1$, we define the *type* $\tau_\rho(J_\rho)$ of J_ρ as an expression showing the (parenthesized) lengths of the cycles composing J_ρ, each one subindexed with its corresponding y. In the case of our 4 examples, we have:

$$\tau_2(J_2) = (3_1),\ \tau_3(J_3) = (7_4),\ \tau_4(J_4) = (15_{12}) \text{ and } \tau_5(J_5) = (21_{16})(7_3)(3_1).$$

Let us see how the *ds* notation given above can be extended to the elements $p(Q, a)$ of $V(H_\rho)$, as in Subsection 4.1. We express the two-level expressions $\substack{v\\u}$ for the $(\mathbf{P}_2^{\rho-2}, 1)$-permutations $v = p(\mathbf{P}_2^{\rho-2}), 1)$ as follows, for $\rho = 3, 4$:

$$\begin{array}{lcl|lcl} v & = & 123(45)(67) & v & = & 1234567(89)(ab)(cd)(ef) \\ u & = & 123(11)(11) & u & = & 1234567(11)(11)(11)(11) \end{array}$$

where: **(a)** each fixed point of v is repeated in u under its appearance in v; **(b)** *ds*-companion x-cycles are well-defined cycles only if $x > 2$ and **(c)** we say that each transposition (a_0a_1) in v has *degenerate ds-companion cycle* $(b\,b)$, (in fact, not a well-defined cycle, just notation), where ba_0a_1 is a line of $\mathbf{P}_2^{\rho-1}$. We also say that the pivot b *dominates* each (a_0a_1) in v.

We define now the *types* of the $(\mathbf{P}_2^{\rho-2}, 1)$-permutations $p(\mathbf{P}_2^{\rho-2}, 1)$ above as:

$$\begin{array}{c} \tau_3(p(\mathbf{P}_2^1, 1)) = \tau_3(123(45)(67)) = (1(2)(2)) = (1((2)^2)) \\ \tau_4(p(\mathbf{P}_2^2, 1)) = \tau_4(1234567(89)(ab)(cd)(ef)) = (1((2)^4)) \end{array}$$

with the *domination* expressed, for each $p(\mathbf{P}_2^{\rho-2}, 1)$, by a pair of parentheses containing the length $= 1$ of the pivot $b = 1$ followed by the parenthesized lengths of the cycles it dominates. More generally, if v is of the form $p(Q, a)$ in $V(H_\rho)$, then we take $\tau_\rho(v) = (1((2)^{2^{\rho-2}}))$.

This concept of domination will permit us to extend the initiated notion of type of an $\mathcal{A}$-permutation. For example, the doubling provided by the embeddings $\Psi_\rho : V(H_{\rho-1}) \to V(H_\rho)$ in Subsection 4.1 allows the expression of other types of $\mathcal{A}$-permutations, from Subsections 4.5 on. For now, we define the type of $I_\rho = 12\dots(2^\rho - 1) = 12\dots m_1$ to be $\tau_\rho(I_\rho) = (1)$.

The following fact is used in counting $\mathcal{A}$-permutations and finding the diameter of H_ρ.

Theorem 4.1 *The distance $d(v, I_\rho)$ in H_ρ from an $\mathcal{A}$-permutation v to the identity I_ρ is related to the cardinality of the fixed-point set F_v of v in $\mathbf{P}_2^{\rho-1}$ by*

$$\log_2(1 + |F_v|) + d(v, I_\rho) = \rho. \tag{1}$$

Proof. I_ρ has $|F_{I_\rho}| = 2^\rho - 1 = m_1$, so (1) holds for I_ρ because $\log_2(1 + (2^\rho - 1)) = \rho$. Adjacent to I_ρ are the elements of the form $p(Q, a)$, each of which has $2^{\rho-1} - 1$ fixed points, so (1) holds for the vertices at distance 1 from I_ρ. Successively, the vertices at distance 2 from I_ρ have $2^{\rho-2} - 1$ fixed points, and so on, inductively, until the $\mathcal{A}$-permutations in $V(H_\rho)$ have no fixed points, (J_ρ included), and are at distance ρ from I_ρ, so they satisfy (1), too. □

4.4 Two-line notation for J_ρ

Another way to look at J_ρ is in its two-line, or relation, notation:

$$J_\rho = \binom{\xi_\rho}{\eta_\rho} = \binom{123}{312}, \binom{1234567}{3475612}, \binom{123456789abcdef}{3478bcfde9a5612}, \binom{123456789abcdefghijklmnopqrstuv}{3478bcfgjknorsvtupqlmhide9a5612},$$

for $\rho = 2, 3, 4, 5$, respectively. The lower levels here, that we call levels η_ρ, have the following pattern. Each symbol pair in the following list L:

$$12,\ 34,\ 56,\ \dots,\ (2i-1)(2i),\ \dots,\ (2^{r-1}-3)(2^{r-1}-2),$$

is alternatively placed in the level η_ρ, below the $2^{\rho-1}$ position pairs $(2i-1)(2i)$ of contiguous points $\neq 2^{\rho-1}$, according to the following instructions: **(a)** place the starting pair of L in the rightmost pair of still-empty positions of η_ρ and erase it from L; **(b)** place the resulting new starting pair of L in the leftmost pair of still-empty positions of η_ρ and erase it from L; **(c)** repeat (a) and (b) alternatively until the point $m_1 = 2^\rho - 1$ is left alone in L; **(d)** place $m_1 = 2^\rho - 1$ in the $(2^{\rho-1}-1)$-th position of η_ρ, that still remained empty. Now, η_ρ looks like:

$$3478\dots(4i-1)(4i)\dots(2^{r-1}-5)(2^{r-1}-4)(2^{r-1}-1)(2^{r-1}-3)(2^{r-1}-2)\dots(4i+1)(4i+2)\dots5612.$$

and can be expressed by means of the function f defined by:

$$\begin{array}{lll}
f(2i) & = 4i, & (i{=}1,\dots,2^{\rho-2}-1); \\
f(2i-1) & = 4i-1, & (i{=}1,\dots,2^{\rho-2}-1); \\
f(2^\rho-2i+1) & = 4i+2, & (i{=}1,\dots,2^{\rho-2}); \\
f(2^\rho-2i) & = 4i+1, & (i{=}1,\dots,2^{\rho-2}); \\
f(2^{\rho-1}-1) & = 2^\rho-1. &
\end{array}$$

4.5 The other types at distance ρ from I_ρ

The leftmost $2^{\rho-1}-1$ symbols of η_ρ in Subsection 4.4 form a $(\rho-2)$-subspace ζ_ρ of $\mathbf{P}_2^{\rho-1}$. Let $z(j) = p(\zeta_\rho, f(j)) \in V(H_\rho)$, with fixed-point set ζ_ρ and pivot $f(j) \in \zeta_\rho$, where $j = 1, \dots, 2^{\rho-1}-1$.

The products $J_\rho.z(j)$, $(j = 2, 4, 6, \dots,\ 2^{\rho-1}-2)$, yielding each a permutation $w_\rho(j) = J_\rho.z(j)$, are at distance ρ from I_ρ and produce pairwise different new types. Also, successive powers of these permutations $w_\rho(j)$ must be checked, in order to extract any remaining types at distance ρ from I_ρ. We exemplify these observations for $r = 3, 4, 5$.

First, $w_3(2) = J_3.z(2) = (1372456).437(15)(26) = (1376524)$, which is a 7-cycle with *ds*-companion cycle switched two positions to the right, that we indicate by defining type $\tau_3(w_3(2)) = (7_2)$. Summarizing this, we have:

$w_3(2)$	(1376524)
$ds{-}level$	(2413765)
$type$	(7_2)

Moreover, $\tau_3(w_3(2)) = \tau_3((w_3(2))^2) = \dots = \tau_3((w_3(2))^6) = (7_2)$, but $(w_3(2))^7$ is the identity permutation, whose type is (1). So, taking powers of $w_3(2)$ did not contribute any new types.

For $\rho = 3$, an extension of τ_ρ takes place, in which the domination of a transposition by its pivot extends to the domination of a cycle by another cycle, shown parenthesized as in Subsections 4.2-3. (More examples in Subsection 4.6). A special case, present in the remaining examples of this section, is via a c_1-cycle C_1 dominating a c_2-cycle C_2 which in turn dominates a c_3-cycle C_3, and so on, until a c_x-cycle C_x dominates C_1, so that a *super-cycle* $(C_1, C_2, \dots, C_x)$ appears. The type of the resulting permutation (or permutation factor) is taken as $(c_1(c_2(c_3(\dots(c_x(_y)\dots)))))$, where y, appearing as a subindex between the innermost parentheses, is obtained by aligning C_1, C_2, $\dots$, C_x and their respective *ds*-companion cycles D_1, D_2, $\dots$, D_x so that each dominated *ds*-companion cycle D_{i+1} is presented in the same order as its dominating cycle C_i, for $i = 1, \dots, x$. In this disposition, y is the shift of the *ds*-companion cycle of C_1 with respect to its dominating cycle C_x.

For example, the values of $w_4(j)$ and the types $\tau_4(w_4(j))$, for $j = 2, 4, 6$, are as follows:

j	2	4	6
$w_4(j)$	$(5be)(2489ad)(137f6c)$	$(2485b)(137fa)(cde96)$	$(137feda5b6c9248)$
$ds{-}level$	$(e5b)(6c137f)(2489ad)$	$(6cde9)(2485b)(137fa)$	$(248137feda5b6c9)$
$type$	$(3_1)(6(6(_0)))$	$(5(5(5(_1))))$	(15_3)

Powers of $w_4(2)$ yield new types:

i	2	3
$(w_4(2))^i$	$(5eb)(28a)(49d)(176)(3fc)$	$5(8d)(36)b(29)(7c)e(1f)(4a)$
$ds{-}level$	$(b5e)(a28)(d49)(617)(c3f)$	$5(55)(55)b(bb\ )(bb\ )e(ee\ )(ee\ )$
$type$	$(3_1)^5$	$(1((2)^2))^3$

The type $(3_1)^5$ here still represents a permutation at maximum distance $= 4$ from I_4. However, the type $(1((2)^2))^3$ has distance 2 from I_4. Subsequent powers of these $w_4(j)$'s, $(j = 2, 4, 6)$, do not yield new types of elements of $V(H_4)$.

We present the $\mathcal{A}$-permutations $w_5(2i)$, $(1 \le i \le 6)$, and their types:

$$\begin{array}{lcl|l}
w_5(2) & = & (137fv6co9ju5bnmlit248gpakhqdres) & \tau_5(w_5(4))= \ (31_{19}); \\
w_5(4) & = & (137fvaktesdr248glu9jihmp6co5bnq) & \tau_5(w_5(2))= \ (31_{13}); \\
w_5(6) & = & (137fves9jmtakp248ghilq5bnudr6co) & \tau_5(w_5(8))= \ (31_{18}); \\
w_5(8) & = & (137fvi9jak5bn248gdrqpuhesl6cotm) & \tau_5(w_5(6))= \ (31_{17}); \\
w_5(10) & = & (137fvm5bn6copqtidrul248g9jeshak) & \tau_5(w_5(12))= \ (31_{12}); \\
w_5(12) & = & (137fvqh6colestupm9j248g5bnakdri) & \tau_5(w_5(10))= \ (31_{11});
\end{array}$$

no new types are obtained from these $w_5(2i)$'s by considering their powers.

4.6 Types at distances $< \rho$ from I_ρ

We notice that: **(a)** for $j = 1, 3, \dots, 2^{\rho-1} - 1$, the elements $w_\rho(j) = J_\rho.z(j)$ of $V(H_\rho)$ are at distance $\rho - 1$ from I_ρ and provide pairwise different new types; **(b)** if successive powers of these $w_\rho(j)$'s are taken, they must be at distances $< \rho - 1$ from I_ρ and may provide new types of $\mathcal{A}$-permutations. We exemplify these observations for $\rho = 3, 4, 5$. First, we have:

j	1	3
$w_3(j)$	5(246)(137)	6(24)(1375)
$ds{-}level$	5(624)(246)	6(66)(2424)
$type$	$(3_1(3))$	(1(2(4)))

The square of $w_3(1)$ still preserves its type. However, $(w_3(3))^2 = 624(17)(35) = p(624, 6)$. Thus, $\tau_3((w_3(3))^2) = (1((2)^2))$. Also, it can be seen that $w_4(2i + 1)$ has types

$$(1(2(4((4)^2)))), \quad (7_3(7)), \quad (7_5(7)), \quad (1(2))(3_1((3)(6))),$$

for $i = 0, 1, 2, 3$, respectively. By taking the squares of these permutations, we get that $(w_4(3))^2$ and $(w_4(5))^2$ preserve the respective types of $w_4(3)$ and $w_4(5)$, while it is seen that the types of $w_4(1)$ and $w_4(7)$ are

$$(1((2)^2))^3, \quad (3_1((3)^3)),$$

respectively, the first one of which was seen in Subsection 4.5. Finally, it can be seen that $w_5(2i + 1)$ has types

$$\begin{array}{llll}
(5((5)(5((5)(5(5)(_1)))))), & (1(2))(7_4(7(14))), & (1(2(4)))(3(3(6(12)))), & (15_3(15)), \\
(1(2))(7_2(7(14))), & (3(3))(6((6)(6(6)))), & (15_{11}(15)), & (1(2(4(8)4(8))),
\end{array}$$

for $i = 0, \dots, 7$, respectively.

4.7 A set of $V(H_{\rho-1})$-coset representatives in $V(H_\rho)$

The objective of this subsection is to establish a set of representatives of the cosets of $V(H_\rho)$ mod $V(H_{\rho-1})$, which we do here for $\rho \le 5$ and conjecture for $\rho > 5$. First, we define a type $\tau'_\rho = \tau_\rho(v)$ of certain vertices $v \in V(H_\rho)$:

$$\begin{array}{ll|ll}
\tau'_2 & =(1(2)), & \tau'_6 & =(1(2(4((4(8(16)))^2)))), \\
\tau'_3 & =(1(2(4))), & \tau'_7 & =(1(2(4((4(8(16((16)^2)))^2))))), \\
\tau'_4 & =(1(2(4((4)^2)))), & \tau'_8 & =(1(2(4((4(8(16((16(32))^2)))^2))))), \\
\tau'_5 & =(1(2(4((4(8))^2)))) & \tau'_9 & =(1(2(4((4(8(16((16(32(64)))^2)))^2))))), \\
 & & \cdots & =\dots\dots \\
 & & & \\
\tau'_{3s-1} & =(1(2(\dots(2^{2s-1})\dots))), & \tau'_{3s+1} & =(1(2(\dots(2^{2s-1}(2^{2s}((2^{2s})^2)))\dots))), \\
\tau'_{3s} & =(1(2(\dots(2^{2s-1}(2^{2s}))\dots))), & \tau'_{3s+2} & =(1(2(\dots(2^{2s-1}(2^{2s}((2^{2s}(2^{2s+1}))^2)))\dots))),
\end{array}$$

for any $s > 0$. The claimed representatives of cosets of $V(H_\rho)$ mod $V(H_{\rho-1})$ are set as follows, in 5 different categories (a)-(e), where each category (b) and (d) admits two subcategories subindexed α and β, and (Q, a)-permutations $p(Q, a)$ are as in Subsection 4.1:

(a) the identity permutation I_ρ;

(b_α) the permutations $p(\mathbf{P}_2^{\rho-2}, a)$, where $a \in \mathbf{P}_2^{\rho-2}$; e.g.

ρ=3	$123(45)(67)$	ρ=4	$1234567(89)(ab)(cd)(ef)$	$5123467(8d)(9c)(af)(be)$
	$231(46)(57)$		$2134567(8a)(9b)(ce)(df)$	$6123457(8e)(9f)(ac)(bd)$
	$312(47)(56)$		$3124567(8b)(9a)(cf)(de)$	$7123456(8f)(9e)(ad)(bc)$
			$4123567(8c)(9d)(ad)(bf)$	

(b_β) those $p(Q, a)$'s for which Q is a $(\rho - 2)$-subspace containing $a = m_1 = 2^\rho - 1$; e.g.

ρ=3	$716(25)(34)$	ρ=4	$f123cde(4b)(5a)(69)(78)$	$f2578ad(1e)(3c)(4b)(69)$
	$725(16)(34)$		$f145abe(2d)(3c)(69)(78)$	$f3478bc(1e)(2d)(5a)(69)$
	$734(16)(25)$		$f16789e(2d)(3c)(4b)(5a)$	$f3569ac(1e)(2d)(4b)(78)$
			$f2469bd(1e)(3c)(5a)(78)$	

(c) those $p(Q, a)$'s for which $Q \subset \mathbf{P}_2^{\rho-1}$ is a $(\rho - 2)$-subspace containing $a = m_1 - x$, with $x \in \mathbf{P}_2^{\rho-2}$; e.g.

ρ=3	$415(26)(37)$	ρ=4	$81239ab(4c)(5d)(6e)(7f)$	$81459cd(2a)(3b)(6e)(7f)$
	$514(27)(36)$		$91238ab(4d)(5c)(6f)(7e)$	$91458cd(2b)(3a)(6f)(7e)$
	$624(17)(35)$		$a12389b(4e)(5f)(6c)(7d)$	$c14589d(2e)(3f)(6a)(7b)$
	$426(15)(37)$		$b12389a(4f)(5e)(6d)(7c)$	$d14589c(2f)(3e)(6b)(7a)$
	$536(14)(27)$		$\ldots$	$\ldots$
	$635(17)(24)$		$\ldots$	$\ldots$

(d_α) an $\mathcal{A}$-permutation α_ρ of type τ'_ρ selected as follows for each $(\rho - 3)$-subspace X_ρ of $\mathbf{P}_2^{\rho-2}$ and each $x_\rho \in \mathbf{P}_2^{\rho-2} \setminus \bar{X}_\rho$, ($\bar{X}_\rho = \{m_1 - x_3 : x_\rho \in X_\rho\}$): take the fixed point of α_ρ as the smallest point in $\bar{X}_\rho$; take the 2-cycle of α_ρ, with $(m_1 - x_\rho)$ as ds, containing $(m_1 - x_\rho)$ and dominating a 4-cycle containing m_1; subsequent pairs, quadruples, $\ldots 2^s$-tuples $\ldots$ of intervening 4-cycles, 8-cycles, $\ldots, 2^{s+1}$-cycles, $\ldots$, respectively, should have the first 2^{s+1}-cycle ending with the smallest available point of X_ρ, for $s = 1, 2, \ldots$; e.g.

X_3	$\bar{X}_3$	x_3	α_3	X_4	$\bar{X}_4$	x_4	α_4
1	6	4	$6(42)(7315)$	123	edc	8	$c(84)(f73b)(a521)(69ed)$
1	6	5	$6(53)(7214)$	edc	123	9	$c(95)(f63a)(b421)(78ed)$
2	5	4	$5(41)(7326)$	123	edc	a	$c(a6)(f539)(8721)(4bed)$
2	5	6	$5(63)(7124)$	123	edc	b	$c(b7)(f438)(9621)(5aed)$
3	4	5	$4(51)(7236)$	145	eba	8	$a(82)(f75d)(c341)(69eb)$
3	4	6	$4(62)(7135)$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

(d_β) the inverse permutations of the α_ρ's from item d_α;

(e) a total of $(2^{\rho-1} - 1)(2^{\rho-2} - 1)$ $\mathcal{A}$-permutations ξ of type τ'_ρ with: fixed point $\in \mathbf{P}_2^{\rho-2}$; 2-cycle containing m_1; leftmost dominating 4-cycle η starting: at the smallest available point, for $2^{\rho-3}$ of these ξ's, if $\rho \geq 3$; at the next smallest available point, for $2^{\rho-4}$ of the remaining ξ's, not yet used in the η's, if $\rho \geq 4$, etc.; remaining dominated 4-cycles, 8-cycles, etc., if applicable, varying with the next available smallest points; e.g.

ρ=3	$1(76)(2435)$	ρ=4	$1(fe)(2d3c)(46b8)(57a9)$	$2(fd)(1e3c)(45b8)(679a)$
	$2(75)(1436)$		$1(fe)(2d3c)(649a)(758b)$	$2(fd)(1e3c)(54a9)(768b)$
	$3(74)(1526)$		$1(fe)(4b5a)(26d8)(37c9)$	$2(fd)(4b69)(15e8)(37ca)$
			$\ldots$	$\ldots$

The representatives of the cosets of $V(H_\rho)$ mod $V(H_{\rho-1})$ presented above will be called the *selected coset representatives* of $V(H_\rho)$.

Theorem 4.2 *The $\mathcal{A}$-permutations in a fixed category $x \in \{(a), \ldots, (e)\}$ are in 1-1 correspondence with the cosets of $V(H_\rho)$ they determine mod $V(H_{\rho-1})$, at least for $\rho \leq 5$. Thus, they can effectively be referred without confusion as the* selected coset representatives *of $V(H_\rho)$. Moreover, any such coset has the same number $N_\rho(x)$ of $\mathcal{A}$-permutations in each type τ_ρ. Thus, the distribution of types in a coset of $V(H_\rho)$ mod $V(H_{\rho-1})$ generated by an $\mathcal{A}$-permutation in x depends solely on x.*

Proof. The selection of the 5 categories (a)-(e) is effective for producing specific representatives of distinct classes of $V(H_\rho)$ mod $V(H_{\rho-1})$, because the symbol $m_1 = 2^\rho - 1$ is placed once in each strategic position, while setting the remaining entries and difference symbols to yield tightly different situations, and yet covering each coset just once. On the other hand, the representatives in each category are equivalent with respect to the structure of the cosets of $\mathbf{P}_2^{\rho-1}$ mod $\mathbf{P}_2^{\rho-2}$ that yields the classes of $V(H_\rho)$ mod $V(H_{\rho-1})$. Thus, each of these cosets has the same number of representatives, in particular in each type τ_ρ. $\square$

4.8 Order and diameter of G_r^σ via simplified types

The *simplified type* $\gamma_\rho(v)$ of an $\mathcal{A}$-permutation v of $V(H_\rho)$ is defined by writing from left to right the parenthesized cycle lengths of $\tau_\rho(v)$ in non-decreasing order, (no dominating parentheses or subindices now), with the cycle-length multiplicities $\mu > 1$ expressed via external superscripts. This will allow us to write the cycle lengths of the simplified types $\gamma'_\rho = \gamma_\rho(v)$ corresponding to the types $\tau'_\rho = \tau_\rho(v)$ of Subsection 4.7 as products of prime powers between parentheses that distinguish the resulting exponents from the external multiplicity superscripts. For the identity permutation, we agree that $\gamma_\rho(I_\rho) = \gamma_\rho(1\dots(2^\rho-1)) = \gamma_\rho(1\dots m_1) = (1)$.

We present tables that exemplify the assertion of Theorem 4.2 by means of simplified types, for $\rho = 2,3,4,5$:

	D	(a)	(b)	(c)	(d)	(e)	Σ_{row}
γ_2	$N_2(x)$	1	2	–	–	–	3
(1)	0	1	–	–	–	–	1
(2)	1	1	1	–	–	–	3
(3)	2	–	1	–	–	–	2
	Σ_{col}	2	2	–	–	–	6
γ_3	$N_3(x)$	1	6	6	12	3	28
(1)	0	1	–	–	–	–	1
$(2)^2$	1	3	2	1	–	–	21
$(3)^2$	2	2	2	1	3	–	56
$(2)(4)$	2	–	2	2	1	2	42
(7)	3	–	–	2	2	4	48
	Σ_{col}	6	6	6	6	6	168
γ_4	$N_4(x)$	1	14	28	56	21	120
(1)	0	1	–	–	–	–	1
$(2)^4$	1	21	4	1	–	–	105
$(2)^6$	2	–	6	3	–	2	210
$(2)^2(4)^2$	2	42	30	12	6	6	1260
$(3)^4$	2	56	24	6	10	–	1120
$(2)(4)^3$	3	–	24	24	18	24	2520
$(2)(3)^2(6)$	3	–	32	26	30	24	3360
$(7)^2$	3	48	48	48	48	48	5760
$(3)(6)^2$	4	–	–	12	18	16	1680
$(5)^3$	4	–	–	12	12	16	1344
(15)	4	–	–	24	24	32	2688
$(3)^5$	4	–	–	–	2	–	112
	Σ_{col}	168	168	168	168	168	20160
γ_5	$N_5(x)$	1	30	120	240	105	496
(1)	0	1	–	–	–	–	1
$(2)^8$	1	105	8	1	–	–	465
$(2)^{12}$	2	210	84	21	–	12	6510
$(2)^4(4)^4$	2	1260	308	56	28	20	26040
$(3)^8$	2	1120	224	28	36	–	19840
$(2)^2(4)^6$	3	2520	1848	672	504	504	312480
$(2)^2(3)^4(6)^2$	3	3360	2464	812	756	756	416640
$(7)^4$	3	5760	2688	896	896	640	476160
$(2)^6(4)^4$	3	–	504	210	84	168	78120
$(3)^2(6)^4$	4	1680	1680	1512	1848	1488	833280
$(5)^6$	4	1344	1344	1344	1344	1344	666624
$(15)^2$	4	2688	2688	2688	2688	2688	1333248
$(3)^{10}$	4	112	112	56	168	48	55552
$(2)(7)^2(14)$	4	–	3072	3072	2688	3072	1428480
$(2)(4)^3(8)^2$	4	–	1344	1344	1176	1344	624960
$(2)(3)^2(4)(6)(12)$	4	–	1792	1624	1736	1600	833280
$(3)(7)(21)$	5	–	–	1792	2176	2048	952320
(31)	5	–	–	4032	4032	4608	1935360
	Σ_{col}	20160	20160	20160	20160	20160	9999360

In those tables, the header row indicates: first, a column citing the different existing simplified types γ_ρ; second, a column for the common distance D of the $\mathcal{A}$-permutations of each of these γ_ρ's to I_ρ, according to Theorem 4.1; then, a column for each $x \in \{(a),\dots,(e)\}$; and finally, a column Σ_{row} to be explained below; the second, auxiliary, row indicates the number $N_\rho(x)$ of cosets (as in Theorem 4.2) in each category x; each remaining row, but for the last one, contains, in column x, the number of selected coset representatives of

$V(H_\rho)$ in x with a specific simplified type γ_ρ, so it is denoted row_{γ_ρ}; the final column Σ_{row} contains in row_{γ_ρ} the scalar product of the 5-vectors

$$(row_{\gamma_\rho}(a), row_{\gamma_\rho}(b), row_{\gamma_\rho}(c), row_{\gamma_\rho}(d), row_{\gamma_\rho}(e)) \quad \text{and} \quad (N_\rho(a), N_\rho(b), N_\rho(c), N_\rho(d), N_\rho(e));$$

the sum of the values of column Σ_{row} yields the order of H_ρ, placed in the lower-right corner.

The doubling provided by the embeddings $\Psi_\rho : V(H_\rho) \to V(H_\rho)$ of Subsection 4.1 happens in several places in these tables. If we indicate by ψ_ρ the map induced by Ψ_ρ at the level of simplified types, then we have: $\psi_3((2)) = (2)^2$, $\psi_3((3)) = (3)^2$, etc. In fact, all the simplified types of $V(H_\rho)$ appear squared in $V(H_\rho)$.

The $\mathcal{A}$-permutations of type τ'_ρ yield simplified types γ'_ρ as follows:

$$\begin{array}{ll|ll}
\gamma'_2 & =(2), & \gamma'_6 & =(2)(4)^3(8)^2(16)^2, \\
\gamma'_3 & =(2)(4), & \gamma'_7 & =(2)(4)^3(8)^2(16)^6, \\
\gamma'_4 & =(2)(4)^3, & \gamma'_8 & =(2)(4)^3(8)^2(16)^6(32)^4, \\
\gamma'_5 & =(2)(4)^3(8)^2, & \gamma'_9 & =(2)(4)^3(8)^2(16)^6(32)^4(64)^4, \\
 & & \cdots & =\ldots\ldots \\
 & & & \\
\gamma'_{s+1} & =(\gamma'_s)(2^{2s})^s, & \gamma'_{s+3} & =(\gamma'_s)(2^{2s})^{3s}(2^{2s+1})^{2s}, \\
\gamma'_{s+2} & =(\gamma'_s)(2^{2s})^{3s}, & \gamma'_{s+4} & =(\gamma'_s)(2^{2s})^{3s}(2^{2s+1})^{2s}(2^{2s+2})^{2s},
\end{array}$$

for $s \equiv 2 \bmod 4$.

Theorem 4.3 *Let* $V_\rho = \Pi_{i=1}^{\rho-2}(2^{i-1}(2^i-1))$ *and let* $N'_\rho(x)$ *be the number of selected coset representatives of* $V(H_\rho)$ *with simplified type* γ'_ρ *in category* $x \in \{(a), \ldots, (e)\}$*. Then, for* $2 < \rho$ *and at least for* $\rho \leq 5$*, it holds that:* **1.** $N'_\rho(a) = 0$*;* **2.** $N'_\rho(b) = N'_\rho(c) = N'_\rho(e) = 2^{\rho-2}V_\rho$; **3.** $N'_\rho(d) = (2^{\rho-2}-1)V_\rho$.

Proof. The corollary follows by inductively counting the selected coset representatives of $V(H_\rho)$ with simplified type γ'_ρ in categories (a)-(e), starting from its values in the given tables, for $\rho = 2, 3, 4, 5, \ldots$ □

Corollary 4.4 *Categories* (a)-(e) *are composed by the selected pairwise disjoint coset representatives of* $V(H_\rho)$*, which yields a partition of* $V(H_\rho)$ *mod* $V(H_{\rho-1})$*, at least for* $\rho \leq 5$*.*

Proof. For $\rho > 2$, the corollary follows from Theorem 4.2 with distribution as in Theorem 4.3 for the vertices of type τ'_ρ, or simplified type γ'_ρ. The corollary also holds for $\rho = 2$.

The statement can be checked out alternatively by means of the $\mathcal{A}$-permutation $(J_{\rho-1})^2$ (obtained via the doubling of $J_{\rho-1}$ in $V(H_\rho)$, Subsection 4.1) and the coset representatives of $V(H_\rho)$ selected with the type of $(J_{\rho-1})^2$, yielding alternative simplified types $\gamma''_2 = (3)^2$, $\gamma''_3 = (7)^2$, $\gamma''_4 = (15)^2$, $\gamma''_5 = ((3)(7)(21))^2, \ldots$ In this case, by defining $N''_\rho(x)$ as $N'_\rho(x)$ was in Theorem 4.3, but with γ''_ρ instead of γ'_ρ, we get uniformly that $N''_\rho(x) = 2^{\rho-2}V_\rho$, where $x \in \{(a), \ldots, (e)\}$. This covers all the classes of $V(H_\rho)$ mod $V(H_{\rho-1})$ and again implies the statement. □

Theorem 4.5 *With the notation of* Theorem 4.3, $|V(H_\rho)| = V_{\rho+2}$ *at least for* $\rho \leq 5$*. Moreover, the following properties of the graphs* G^σ_r *hold for* $\sigma \geq 1$*,* $\rho \geq 2$ *and at least for* $\rho \leq 5$*:*
(A) $|V(G^\sigma_r)|$ *is as asserted in* Remark **c** *of* Section 2*;*
(B) G^σ_r *is* $sm_1(t-1)$*-regular;*
(C) *The diameter of* G^σ_r *is* $\leq 2r-2$*.*
Thus, order, degree and diameter of G^σ_r *are respectively:* $O(2^{(r-1)^2}), O(2^{r-1})$ *and* $O(r-1)$*.*

Proof. Item (C) is a corollary of Theorem 4.1. Item (B) can be deduced from the definition of G^σ_r. Recall that $N_\rho(x)$ is the number of cosets (as in Theorem 4.2) in each category $x \in \{(a), \ldots, (e)\}$. Counting cosets obtained via doubling (Subsection 4.1) in each category shows that at least for $\rho \leq 5$: **(a)** $N_\rho(a) = 1$; **(b)** $N_\rho(b) = 2(2^{\rho-1}-1)$; **(c)** $N_\rho(c) = 2^{\rho-2}(2^{\rho-1}-1)$; **(d)** $N_\rho(d) = 2N_\rho(c)$; **(e)** $N_\rho(e) = (2^{\rho-2}-1)(2^{\rho-1}-1)$. Each coset in these categories contains exactly $|V(H_{\rho-1})|$ $\mathcal{A}$-permutations. Thus, $|V(H_\rho)| = V_{\rho+2}$. Since G^σ_r is the disjoint union of $\binom{r}{\sigma}_2$ copies of $T_{ts,t}$, item (A) follows. Finally, we would get that $|V(G^\sigma_r)| = O(2^{(r-1)^2})$, since $(2^r-1) \leq \binom{r}{\sigma}_2$ and $|V(G^1_r)| = (2^r-1)\Pi_{i=2}^{r-1}(2^{i-1}(2^i-1)) = O(2^{r-1}4^{1+2+3+\ldots+(r-2)}) = O(2^{r-1+(r-2)(r-1)})$, which is $O(2^{(r-1)^2})$. □

5 $\vec{\mathcal{C}}$-homogeneity of G_r^σ

Theorem 5.1 *G_r^σ is a connected $m_1s(t-1)$-regular homogeneous $\{\vec{T}_{ts,t}\}_{\ell_1}^{m_1}\{\vec{K}_{2s}\}_{\ell_2}^{m_2}$-graph which is not ultrahomogeneous unless $(r,\sigma)=(3,1)$, at least for $r \leq 8$ and $\rho \leq 5$. Moreover, $G_r^\sigma = \mathcal{G}_r^\sigma$ if and only if $\sigma = r-2$; in this case, $G_{\sigma+2}^\sigma$ is $\{K_4\}$-ultrahomogeneous. Furthermore, $\mathcal{A}(G_r^\sigma) = \mathcal{N}_r^\sigma \times_\lambda \mathcal{H}_\rho$ with $\lambda : \mathcal{H}_\rho \to \mathcal{A}(\mathcal{N}_r^\sigma)$ given by $\lambda(h_w) = (k \mapsto h_w^{-1}kh_w : k \in \mathcal{N}_r^\sigma)$, with $|\mathcal{N}_r^\sigma|$ as in Theorem 3.1 and $|\mathcal{H}_\rho| = |V(H_\rho)| = V_{\rho+2}$, as in Theorem 4.5.*

Proof. Let τ be an element of $\mathcal{A}(G_r^\sigma)$. Then, τ transforms $\epsilon_r^\sigma = (v_r^\sigma, u_r^\sigma)$ into an arc of G_r^σ. By Section 3 and Subsection 4.1, τ can be presented as $\tau = \psi^\tau.\phi^\tau = \phi^\tau.\psi^\tau$, where ϕ^τ is a permutation of affine σ-subspaces of $\mathbf{P}_2^{r-1}$ and ψ^τ is a permutation of the non-initial entries of ordered pencils that are vertices of G_r^σ, e.g. a permutation of the indices k of entries A_k of such ordered pencils, where $0 < k \leq m_1 = 2^\rho - 1$.

The subgroup of $\mathcal{N}_r^\sigma = \mathcal{A}(N_{G_r^\sigma}(v_r^\sigma))$ fixing the lexicographically smallest neighbor u_r^σ of v_r^σ is formed by the automorphisms τ in item **(A)** of Section 3 with the point $\pi \in \mathbf{P}_2^{\rho-1}$ taken as the third lexicographically smallest such point ($= 3 \times 2^{\sigma-1}$). These automorphisms constitute a subgroup of $\mathcal{N}_r^\sigma$ with $2^{\rho-1}$ elements.

For any two induced copies X_1, X_2 of $T_{ts,t}$ (resp. T_{2k}) in G_r^σ, and arcs v_1w_1, v_2w_2 of X_1, X_2, respectively, there exist automorphisms Φ_1, Φ_2 of G_r^σ such that $\Phi_i(v_r^\sigma) = v_i$ and $\Phi_i(u_r^\sigma) = w_i$, sending $N_{G_r^\sigma}(v_i) \cap X_i$ onto $N_{G_r^\sigma}(v_r^\sigma) \cap X$, for $i = 1, 2$, where X is the lexicographically smallest copy of $T_{ts,t}$ (resp. T_{2k}) in G_r^σ, namely $[(\mathbf{P}_2^\sigma)_1]_r^\sigma$ (resp. $[U]_r^\sigma$, where U is the third lexicographically smallest $(r-1, \sigma-1)$-ordered pencil of $\mathbf{P}_2^{r-1}$, which shares with $[\mathbf{P}_2\sigma_1]_r^\sigma$ just u_r^σ). As a result, the composition $\Phi_2.\Phi_1^{-1}$ in $\mathcal{A}(G_r^\sigma)$ takes X_1 onto X_2, and v_1w_1 onto v_2w_2. This implies that G_r^σ is a homogeneous $\{\vec{T}_{ts,t}\}_{\ell_1}^{m_1}\{\vec{K}_{2s}\}_{\ell_2}^{m_2}$-graph.

Recall from [3] that G_3^1 is $\{K_4, K_{2,2,2}\}$-ultrahomogeneous. Whenever $\rho = 2$, it holds that $\mathcal{G}_r^\sigma = G_r^\sigma$, and this is K_4-ultrahomogeneous by an argument similar to that of [3]. From the remarks in Section 2, it can be seen that, for $(r,\sigma) \neq (3,1)$, there are automorphisms of the copy of $T_{ts,t}$ in G_r^σ that contains the edge $\epsilon_r^\sigma = v_r^\sigma u_r^\sigma$, even fixing v_r^σ and u_r^σ, but they cannot be extended to any automorphism of G_r^σ. A similar conclusion holds for K_{2s}, provided $\sigma \neq r-2$, for which $K_{2s} = K_4$. □

Remark. It can be seen that G_r^σ is the Menger graph [2] of a $(|V(G_r^\sigma)|_{m_2}, (l_2)_{2s})$ configuration whose points and lines are the vertices and the copies of K_{2s} in G_r^σ, respectively. For example, G_4^2 is the Menger graph of a $(210_{12}, 630_4)$ configuration. However, if $\sigma = 1$ then $|V(G_r^\sigma)|_{m_2} = (l_2)_{2s}$, the said configuration is self-dual and the Menger graph coincides with the corresponding dual Menger graph, as is the case of G_3^1, presented in [3].

Conjecture 5.2 *The results of* Theorems 3.1, 4.2-4.5 *and* 5.1 *hold for $r > 8$ and $\rho > 5$.*

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