On certain $\mathcal{C}$-ultrahomogeneous graphs obtained from cubic distance transitive graphs

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Abstract

The notion of a $\mathcal{C}$-ultrahomogeneous (or $\mathcal{C}$-UH) graph due to D. Isaksen et al. is adapted for digraphs and applied to the cubic distance-transitive graphs, considered both as graphs and digraphs, when $\mathcal{C}$ is formed by shortest cycles and $(k-1)$-paths, with k = arc-transitivity. Moreover, $(k-1)$-powers of shortest cycles taken with orientation assignments that make these graphs become $\mathcal{C}$-UH digraphs are 'zipped' into $\mathcal{C}$-UH graphs. In this note, we do this for the Pappus graph and the Desargues. graph

Keywords: ultrahomogeneous graph; digraph; shortest cycle; arc-transitivity

1 Preliminaries

The study of ultrahomogeneous graphs (resp. digraphs) can be traced back to [15], [9], [14], [3] and [11], (resp. [8], [13] and [4]). Following a line of research initiated by [12], given a collection $\mathcal{C}$ of (di)graphs closed under isomorphisms, a (di)graph G is said to be $\mathcal{C}$-*ultrahomogeneous* (or $\mathcal{C}$-UH) if every isomorphism between two induced members of $\mathcal{C}$ in G extends to an automorphism of G. If $\mathcal{C} = \{H\}$ is the isomorphism class of a (di)graph H, we say that such a G is $\{H\}$-UH or H-UH. In [12], $\mathcal{C}$-UH graphs are defined and studied when $\mathcal{C}$ is the collection of either **(a)** the complete graphs, or **(b)** the disjoint unions of complete graphs, or **(c)** the complements of those unions.

We may consider a graph G as a digraph by considering each edge e of G as a pair of oppositely oriented (or O-O) arcs $\vec{e}$ and $(\vec{e})^{-1}$. Then, 'zipping' $\vec{e}$ and $(\vec{e})^{-1}$ allows to recover e, a technique to be used below.

Let M be a sub(di)graph of a (di)graph H and let G be both an M-UH and an H-UH (di)graph. We say that a (di)graph G is (fastened) $(H;M)$-*UH* if, given a copy H_0 of H in G containing a copy M_0 of M, then there exists exactly one copy $H_1 \neq H_0$ of H in G such that $V(H_0) \cap V(H_1) = V(M_0)$ and $A(H_0) \cap \bar{A}(H_1) = A(M_0)$, where $\bar{A}(H_1)$ is formed by those arcs $(\vec{e})^{-1}$ whose orientations are reversed with respect to the orientations of the arcs $\vec{e}$ of $A(H_1)$, and such that no more vertices or arcs than those in M_0 are shared by H_0 and

H_1. The directed case here is used in the constructions of Section 3 and in [6, 7]. In the undirected case, the vertex and arc conditions above can be condensed as $H_0 \cap H_1 = M_0$; this is generalized by saying that an $(H;M)$-UH graph G is an *ℓ-fastened* $(H;M)$-UH graph if given a copy H_0 of H in G containing a copy M_0 of M, then there exist exactly ℓ copies $H_i \neq H_0$ of H in G such that $H_i \cap H_0 = M_0$, for each one of $i = 1, 2, \ldots, \ell$, and such that no more vertices or edges than those in M_0 are shared by each two of $H_0, H_1, \ldots, H_\ell$. We work here and in [6, 7] with the cubic distance-transitive (or CDT) graphs G, [2]:

CDT graph G	n	d	g	k	η	a	b	h	κ
Tetrahedral graph K_4	4	1	3	2	4	24	0	1	1
Thomsen graph $K_{3,3}$	6	2	4	3	9	72	1	1	2
3-cube graph Q_3	8	3	4	2	6	48	1	1	1
Petersen graph	10	2	5	3	12	120	0	0	0
Heawood graph	14	3	6	4	28	336	1	1	0
Pappus graph	18	4	6	3	18	216	1	1	3
Dodecahedral graph	20	5	5	2	12	120	0	1	1
Desargues graph	20	5	6	3	20	240	1	1	3
Coxeter graph	28	4	7	3	24	336	0	0	3
Tutte 8-cage	30	4	8	5	90	1440	1	1	2
Foster graph	90	8	10	5	216	4320	1	1	0
Biggs-Smith graph	102	7	9	4	136	2448	0	1	3

where n, d, g, k, η and a are order, diameter, girth, AT or arc-transitivity, number of g-cycles and number of automorphisms, respectively, with b (resp. h) $= 1$ if G is bipartite (resp. hamiltonian) and $= 0$ otherwise, and κ defined as follows: let P_k and $\vec{P}_k$ be respectively a $(k-1)$-path and a directed $(k-1)$-path (of length $k-1$); let C_g and $\vec{C}_g$ be respectively a cycle and a directed cycle of length g; then (see Theorem 2 below): $\kappa = 0$, if G is not $(\vec{C}_g; \vec{P}_k)$-UH; $\kappa = 1$, if G is planar; $\kappa = 2$, if G is $(\vec{C}_g; \vec{P}_k)$-UH with $g = 2(k-1)$; $\kappa = 3$, if G is $(\vec{C}_g; \vec{P}_k)$-UH with $g > 2(k-1)$.

Given a finite graph H and a subgraph M of H with $|V(H)| > 3$, we say that a graph G is (strongly fastened) SF $(H;M)$-*UH* if there is a descending sequence of connected subgraphs $M = M_1, \ldots, M_{|V(H)|-2} \equiv K_2$ such that: **(a)** M_{i+1} is obtained from M_i by the deletion of a vertex, for $i = 1, \ldots, |V(H)|-3$ and **(b)** G is a $(2^i - 1)$-fastened $(H; M_i)$-UH graph, for $i = 1, \ldots, |V(H)|-2$. Theorem 1 below asserts that every CDT graph is an SF $(C_g; P_k)$-UH graph.

Given a graph C and $0 < k \in \mathbf{Z}$ such that k is at most the diameter of C, recall that the k-power graph C^k of C has $V(C^k) = V(C)$ and that two vertices are adjacent in C^k if and only if they are at distance k in C. Theorem 2 establishes which CDT graphs are $(\vec{C}_g; \vec{P}_k)$-UH digraphs. Elevating the resulting oriented cycles to the $(k-1)$-power enables the construction, in Section 3 and in [7], of fastened $\mathcal{C}$-UH graphs, with $\mathcal{C}$ formed by copies of K_3, K_4, C_7 and $L(Q_3)$, when $\kappa = 3$, via 'zipping' of the O-O induced $(k-1)$-arcs shared (as $(k-1)$-paths) by pairs of O-O g-cycles. In particular, the Pappus (resp. Desargues) graph yields the disjoint union of two copies of the Menger graph of the self-dual (9_3)- (resp. (10_3)-) configuration, [5].

2 (C_g, P_k)-UH properties of CDT graphs

Theorem 1 *Let G be a CDT graph of girth $= g$ and AT $= k$. Then G is an SF $(C_g; P_{i+2})$-UH graph, for $i = 0, 1, \ldots, k-2$. In particular, G is a $(C_g; P_k)$-UH graph and has exactly $2^{k-2}3ng^{-1}$ g-cycles.*

Proof. We have to see that each CDT graph G with girth $= g$ and AT $= k$ is a $(2^i - 1)$-fastened $(C_g; P_{k-i})$-UH graph, for $i = 0, 1, \ldots, k-2$. In fact, each $(k-i-1)$-path $P = P_{k-i}$ of any such G is shared exactly by 2^i g-cycles of G, for $i = 0, 1, \ldots, k-2$. Moreover, each two of these 2^i g-cycles have just P in common. This and a simple counting argument for the number of g-cycles, as cited in the table above, yield the assertions in the statement. □

Theorem 2 *The CDT graphs G of girth $= g$ and AT $= k$ that are not $(\vec{C}_g; \vec{P}_k)$-UH digraphs are the Petersen graph, the Heawood graph and the Foster graph. The remaining nine CDT graphs are $(\vec{C}_g; \vec{P}_k)$-UH.*

Proof. Given a $(\vec{C}_g; \vec{P}_k)$-UH graph G, an assignment of an orientation to each g-cycle of G such that the two g-cycles shared by each $(k-1)$-path receive opposite orientations yields a $(\vec{C}_g; \vec{P}_k)$-*orientation assignment* (or $(\vec{C}_g; \vec{P}_k)$-OA). The collection of η oriented g-cycles corresponding to the η g-cycles of G, for a particular $(\vec{C}_g; \vec{P}_k)$-OA will be called an $(\eta\vec{C}_g; \vec{P}_k)$-OAC.

The graph $G = K_4$ on vertex set $\{1, 2, 3, 0\}$ admits the $(4\,\vec{C}_3; \vec{P}_2)$-OAC $\{(123), (210), (301), (032)\}$. The graph $G = K_{3,3}$ obtained from K_6 (with vertex set $\{1, 2, 3, 4, 5, 0\}$) by deleting the edges of the triangles $(1, 3, 5)$ and $(2, 4, 0)$ admits the $(9\,\vec{C}_4; \vec{P}_3)$-OAC $\{(1234), (3210), (4325), (1430), (2145), (0125), (5230), (0345), (5410)\}$. The graph $G = Q_3$ with vertex set $\{0, \ldots, 7\}$ and edge set $\{01, 23, 45, 67, 02, 13, 46, 57, 04, 15, 26, 37\}$ admits the $(6\,\vec{C}_4; \vec{P}_2)$-OAC $\{(0132), (1045), (3157), (2376), (0264), (4675)\}$.

If $G = Pet$ is the Petersen graph, then G can be obtained from the disjoint union of the 5-cycles $\mu_\infty = (u_0u_1u_2u_3u_4)$ and $\nu_\infty = (v_0v_2v_4v_1v_3)$ by the addition of the edges (u_x, v_x), for $x \in \mathbf{Z}_5$. Apart from the two 5-cycles given above, the other ten 5-cycles of G can be denoted by $\mu_x = (u_{x-1}u_xu_{x+1}v_{x+1}v_{x-1})$ and $\nu_x = (v_{x-2}v_xv_{x+2}u_{x+2}u_{x-2})$, for each $x \in \mathbf{Z}_5$. Then the following sequence of alternating 6-cycles and 2-arcs starts and ends up with opposite orientations: $\mu_2^-(u_3u_2u_1)\mu_\infty^+(u_0u_1u_2)\mu_1^-(u_2v_2v_0)\nu_0^-(v_3u_3u_2)\mu_2^+$, where the upper indices $\pm$ indicate either a forward or backward selection of orientation and each 2-path is presented with the orientation of the previously cited 5-cycle but must be present in the next 5-cycle with its orientation reversed. Thus, Pet cannot be a $(\vec{C}_5; \vec{P}_3)$-UH digraph.

For each positive integer n, let I_n stand for the n-cycle $(0, 1, \ldots, n-1)$, where $0, 1, \ldots, n-1$ are considered as vertices. If $G = Hea$ is the Heawood graph, then G can be obtained from I_{14} by adding the edges $(2x, 5+2x)$, for $x \in \{1, \ldots, 7\}$ where operations are in $\mathbf{Z}_{14}$. The 28 6-cycles of G include the following 7 6-cycles: $\gamma_x = (2x, 1+2x, 2+2x, 3+2x, 4+2x, 5+2x)$, where $x \in \mathbf{Z}_7$. Then the following sequence of alternating 6-cycles and 3-arcs starts and ends up with opposite orientations for γ_0:

$$\gamma_0^+(2345)\gamma_1^-(7654)\gamma_2^+(6789)\gamma_3^-(ba98)\gamma_4^+(abcd)\gamma_5^-(10dc)\gamma_6^+(0123)\gamma_0^-,$$

(where tridecimal notation is used, up to $d = 13$). Thus, Hea cannot be a $(\vec{C}_7; \vec{P}_4)$-UH digraph.

If $G = Pap$ is the Pappus graph, then G can be obtained from I_{18} by adding the edges $(1+6x, 6+6x), (2+6x, 9+6x), (4+6x, 11+6x)$, for $x \in \{0, 1, 2\}$, where operations are mod 18. Then G admits a $(18\,\vec{C}_6; \vec{P}_3)$-OAC formed by the oriented 6-cycles $A_0 = (123456)$, $B_0 = (3210de)$, $C_0 = (34bcde)$, $D_0 = (0165gh)$, $E_0 = (4329ab)$, (where octodecimal notation is used, up to $h = 17$), the 6-cycles A_x, B_x, C_x, D_x, E_x obtained by adding $6x$ mod 18 to (the integer representations of) the vertices of A_0, B_0, C_0, D_0, E_0, where $x \in \mathbf{Z}_3 \setminus \{0\}$, and finally the 6-cycles $F_0 = (23ef89)$, $F_1 = (hg54ba)$, $F_2 = (61idc7)$.

If $G = Dod$ is the dodecahedral graph, then G can be seen as a 2-covering graph of the Petersen graph H, where each vertex u_x, (resp., v_x), of H is covered by two vertices a_x, c_x, (resp. b_x, d_x). This can be done so that a $(12\,\vec{C}_5; \vec{P}_2)$-OAC of G is formed by the oriented 5-cycles $(a_0a_1a_2a_3a_4)$, $(c_4c_3c_2c_1c_0)$ and for each $x \in \mathbf{Z}_5$ also $(a_xd_xb_{x-2}d_{x+1}a_{x+1})$ and $(d_xb_{x+2}c_{x+2}c_{x-2}b_{x-2})$.

If $G = Des$ is the Desargues graph, then G can be obtained from the 20-cycle I_{20}, with vertices $4x, 4x+1, 4x+2, 4x+3$ redenoted alternatively x_0, x_1, x_2, x_3, respectively, for $x \in \{0, \ldots, 4\}$, by adding the edges $(x_3, (x+2)_0)$ and $(x_1, (x+2)_2)$, where operations are mod 5. Then G admits a $(20\,\vec{C}_6; \vec{P}_3)$-OAC formed by the oriented 6-cycles A_x, B_x, C_x, D_x, for $x \in \{0, \ldots, 4\}$, where

$$A_x = (x_0x_1x_2x_3(x+1)_0(x+4)_3), \qquad B_x = (x_1x_0(x+4)_3(x+4)_2(x+2)_1(x+2)_2),$$
$$C_x = (x_2x_1x_0(x+3)_3(x+3)_2(x+3)_1), \qquad D_x = (x_0(x+4)_3(x+1)_0(x+1)_1(x+3)_2(x+3)_3).$$

If $G = Tut$ is Tutte's 8-cage, then G can be obtained form I_{30}, with vertices $5x, 5x+1, 5x+2, 5x+3, 5x+4, 5x+5$ denoted alternatively $x_0, x_1, x_2, x_3, x_4, x_5$, respectively, for $x \in \mathbf{Z}_5$, by adding the edges $(x_5, (x+2)_0)$, $(x_1, (x+1)_4)$ and $(x_2, (x+2)_3)$. Then G admits the $(90\,\vec{C}_8; \vec{P}_5)$-OAC formed by the oriented 8-cycles

$$A^0 = (4_50_00_10_20_30_40_51_0), \quad B^0 = (4_24_34_44_51_01_11_21_3), \quad C^0 = (0_20_30_44_14_02_52_42_3),$$
$$D^0 = (3_33_23_14_44_34_21_31_2), \quad E^0 = (4_51_00_50_44_14_03_50_0), \quad F^0 = (4_50_03_54_02_52_41_11_0),$$
$$G^0 = (1_01_12_42_30_20_10_04_5), \quad H^0 = (2_32_41_11_00_50_40_30_2), \quad I^0 = (0_10_20_30_44_14_21_31_4),$$
$$J^0 = (1_00_50_40_33_23_14_44_5), \quad K^0 = (3_13_20_30_20_10_04_54_4), \quad L^0 = (2_32_42_53_03_13_20_30_2),$$
$$M^0 = (3_54_04_10_40_30_20_10_0), \quad N^0 = (3_53_42_12_01_51_40_10_0), \quad O^0 = (4_24_32_22_13_43_31_21_3),$$
$$P^0 = (4_54_44_34_24_10_40_51_0), \quad Q^0 = (4_04_14_21_31_41_53_02_5), \quad R^0 = (0_10_20_33_23_13_01_51_4),$$

together with those obtained from these 18 8-cycles by adding $x \in \mathbf{Z}_5$ uniformly mod 5 to all subindices. Accordingly, these 8-cycles are denoted $A^x, \ldots, R^x$, where $x \in \mathbf{Z}_5$.

If $G = Fos$ is the Foster graph, then G can be obtained from I_{90}, with vertices $5x, 5x+1, 5x+2, 5x+3, 5x+4, 5x+5$ denoted alternatively x_0, x_1, x_2, x_3, x_4, x_5, respectively, for $x \in \mathbf{Z}_{15}$, by adding the edges $(x_4, (x+2)_1)$, $(x_0, (x+2)_5)$ and $(x_2, (x+6)_3)$. The 90 10-cycles of G include the following 15 10-cycles, where $x \in \mathbf{Z}_{15}$.

$$\phi_x=(x_4x_5(x+1)_0(x+1)_1(x+1)_2(x+1)_3(x+1)_4(x+1)_5(x+2)_0(x+2)_1),$$

Then the following sequence of alternating 10-cycles and 4-arcs:

$$\phi_0^+[1_4]\phi_1^-[3_1]\phi_2^+[3_4]\phi_3^-[5_1]\phi_4^+[5_4]\phi_5^-[7_1]\phi_6^+[7_4]\phi_7^-[9_1]\phi_8^+[9_4]\phi_9^-[b_1]\phi_a^+[b_4]\phi_b^-[d_1]\phi_c^+[d_4]\phi_d^-[0_1]\phi_e^+[0_4]$$

may be continued with ϕ_0^-, of opposite orientation to that of the initial ϕ_0^+, where $[x_j]$ stands for a 3-path starting at the vertex x_j in the previously cited (to the left) oriented 10-cycle. Thus, Fos cannot be a $(\vec{C}_{10};\vec{P}_5)$-UH digraph.

The cases of the Coxeter and Biggs-Smith graphs are treated respectively in the Theorem 2 of [6] and the Theorem 2 of [7]. □

3 'Zipping' the $(k-1)$-powers of g-cycles

Given a CDT graph G with $\kappa=3$, consider the collection $\mathcal{C}_g^{k-1}(G)$ of $(k-1)$-powers of oriented g-cycles in the $(\eta\vec{C}_g;\vec{P}_k)$-OAC of G in the proof of Theorem 2. If $k=3$, then each arc $\vec{e}$ of a member C^2 of $\mathcal{C}_g^2(G)$ is marked by the middle vertex of the 2-arc $\vec{E}$ in C for which $\vec{e}$ stands, while the tail and head of $\vec{e}$ are marked by the tail and head of $\vec{E}$, respectively. This is the case below: we consider the CDT graphs G with $\kappa=k=3$ in order to 'zip' such C^2s along their O-O arc pairs to obtain corresponding graphs $Y(G)$ with $\mathcal{C}$-UH properties. In all these cases, the following sequence of operations is performed:

$$G \;\rightarrow\; (\eta\vec{C}_g;\vec{P}_k)\text{-OAC}(G) \;\rightarrow\; \mathcal{C}_g^{k-1}(G) \;\rightarrow\; Y(G).$$

The CDT graphs G with $\kappa=0$ do not admit the approach suggested in the previous paragraph for their g-cycles lack a $(\vec{C}_g;\vec{P}_k)$-OA; those with $\kappa=1$ admit the approach with $Y(G)=G$ so nothing new is obtained more than a corresponding polyhedral graph (embeddable into the sphere) with faces delimited by g-cycles, namely the tetrahedral, 3-cube and dodecahedral graphs; those with $\kappa=2$ again admit the approach, but since $\kappa/2=k-1$, then $Y(G)=(g-1)G^{k-1}$, the multigraph of multiplicity $g-1$ on the $(k-1)$-th power of G.

If G is either the Pappus graph Pap or the Desargues graph Des, then $\mathcal{C}_6^2(G)$ is formed by triangles conforming a graph $Y(G)$ with just two connected components $Y_1(G)$ and $Y_2(G)$.

Each of $Y_1(Pap)$ and $Y_2(Pap)$ is embeddable in a closed orientable surface T_1 of genus 1, or 1-torus. In fact, Figure 1 shows toroidal cutouts of $Y_1(Pap)$ and $Y_2(Pap)$. Notice that the copies of K_3 in $\mathcal{C}_6^2(G)$ are contractible in T_1. These triangles form two collections $\mathcal{H}_1,\mathcal{H}_2$ of copies y_i^j of K_3 closed under parallel translation, where $y=A,B,C,D,E,F$; $i=0,1,2$ and $j=1,2$, namely: the nine of $\mathcal{H}_1$ ($\mathcal{H}_2$) with horizontal edge below (above) its opposite vertex. There is also a collection $\mathcal{H}_0$ of nine non-contractible copies of K_3 in G, traceable

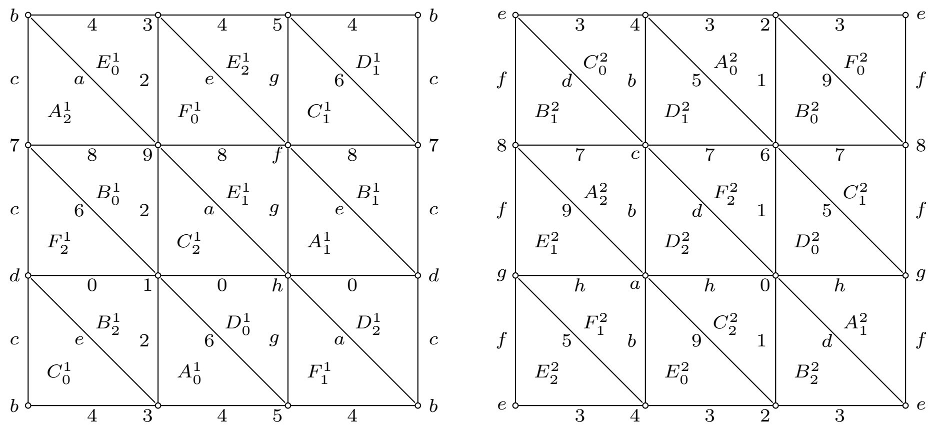

Figure 1: Toroidal cutouts of $Y_1(Pap)$ and $Y_2(Pap)$

linearly in three different parallel directions, three such triangles per direction, with the edges of each triangle marked by an associated common vertex of Pap.

There are embeddings of $Y_1(Pap)$ and $Y_2(Pap)$ in T_1 for which $\mathcal{H}_0$ and either $\mathcal{H}_1$ or $\mathcal{H}_2$ provide the composing faces. In addition, each of $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_0$ is formed by three classes of parallel elements, in the sense that any two of them in such a class do not have vertices in common. The self-dual (9_3)-configuration in the following theorem is the Pappus configuration, [5].

Theorem 3 *$Y_1(Pap)$ and $Y_2(Pap)$ are isomorphic K_2-fastened $\{H_0, H_1, H_2\}$-UH graphs, where H_i is a representative of $\mathcal{H}_i$, for $i = 0, 1, 2$. Moreover, each of $Y_1(Pap)$ and $Y_2(Pap)$ can be taken as the Menger graph of the Pappus self-dual (9_3)-configuration in 12 different forms, by selecting the point set $\mathcal{P}$ and the line set $\mathcal{L} \neq \mathcal{P}$ so that $\{\mathcal{P}, \mathcal{L}\} \subset \{V(Pap), \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2\}$ and the incidence relation either as the inclusion of a vertex in a copy of K_3 or as the containment by a copy of K_3 of a vertex or as the sharing of an edge by two copies of K_3.*

Proof. The statement can be established by managing the data given above. The 12 different claimed forms correspond to the arcs of the complete graph on vertex set $\{V(Pap), \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2\}$. □

If $G = Des$, then $Y_1(G)$ and $Y_2(G)$ are isomorphic K_2-fastened (K_4, K_3)-UH graphs, each formed by five copies of K_4 and ten copies of K_3, with each such copy of K_3: **(a)** not forming part of a copy of K_4 in $Y_1G)$ or $Y_2(G)$; **(b)** having its edges marked by a constant symbol, as shown in Figure 2.

Deleting a copy H of K_4 from such $Y_i(Des)$ yields a copy of $K_{2,2,2}$, four of whose composing copies of K_3, with no common edges, are faces of corresponding copies of $K_4 \neq H$; the other four copies of K_3 are among the ten mentioned copies of K_3 in G. A realization of $Y_1(G)$ (or $Y_2(G)$) in 3-space can be obtained from a regular octahedron O_3 realizing the $K_{2,2,2}$ cited above via the midpoints of the four segments joining the barycenters of four edge-disjoint

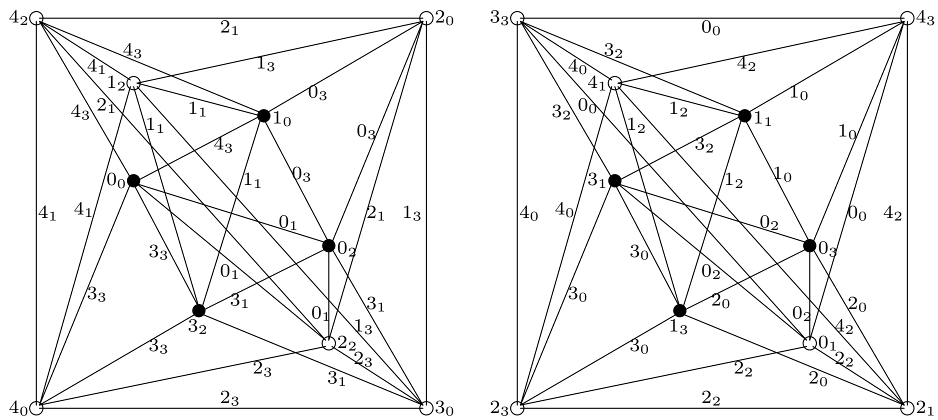

Figure 2: Representations of $Y_1(Des)$ and $Y_2(Des)$

alternate triangles in O_3 to the barycenter of O_3 by constructing the tetrahedra determined by each of these alternate triangles and the nearest constructed midpoint, as well as the fifth central tetrahedron determined by the four midpoints.

By considering the barycenters of the resulting five tetrahedra and the segments joining them, a copy of K_5 in 3-space is obtained. The geometric line graph $L(K_5)$ it gives place to appears as a smaller version of $Y_1(G)$ (or $Y_2(G)$) contained in a octahedron $O'_3 \subset O_3$. This procedure may be repeated indefinitely, generating a sequence of realizations of $Y_1(G)$ (or $Y_2(G)$) in 3-space. Since $Y_1(Des)$ and $Y_2(Des)$ are isomorphic to $L(K_5)$, whose complement is Pet, then this sequence yields a corresponding sequence of realizations of Pet in 3-space.

We notice that the ten vertices and ten copies of K_3 of either $Y_i(Des)$ ($i = 1, 2$) may be considered as the points and lines of the Desargues self-dual (10_3) configuration, and that the Menger graph of this coincides with $Y_i(Des)$, [5]. Each vertex of $Y_i(Des)$ is the meeting vertex of two copies of K_4 and three copies of K_3 not forming part of a copy of K_4.

Theorem 4 *$Y_1(Des)$ and $Y_2(Des)$ are K_2-fastened $\{K_4, K_3\}$-UH graphs composed by five copies of K_4 and ten copies of K_3 each. Moreover, the ten vertices and ten copies of K_3 in either graph constitute the Desargues self-dual (10_3) configuration, which has the graph itself as its Menger graph. Furthermore, both graphs are isomorphic to $L(K_5)$, whose complement is the Petersen graph.* □

Theorem 4 can be partly generalized by replacing $L(K_5)$ by $L(K_n)$ ($n \geq 4$). This produces a K_2-fastened $\{K_{n-1}, K_3\}$-UH graph.

Theorem 5 *The line graph $L(K_n)$, with $n \geq 4$, is a K_2-fastened $\{K_{n-1}, K_3\}$-UH graph with n copies of K_{n-1} and $\binom{n}{3}$ copies of K_3.*

Proof. We assume that each vertex of K_n is taken as a color of edges of $L(K_n)$ under the following rule: Color all the edges between vertices of $L(K_n)$ representing edges incident to a vertex v of K_n with color v. Then, each triple of edge colors for $L(K_n)$ corresponds to the edges of a well determined copy of K_3 in $L(K_n)$. Thus, there are exactly $\binom{n}{3}$ copies of K_3 intervening in $L(K_n)$ looked upon as a $\{K_{n-1}, K_3\}$-UH graph. $\square$

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