On certain $C$-ultrahomogeneous graphs obtained from cubic distance transitive graphs

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Abstract
The notion of a $C$-ultrahomogeneous (or $C$-UH) graph due to D. Isaksen et al. is adapted for digraphs and applied to the cubic distance-transitive graphs considered both as graphs and digraphs when $C$ is formed by shortest cycles and $(k - 1)$-paths with $k = \text{arc-transitivity}$. Moreover, $(k - 1)$-powers of shortest cycles taken with orientation assignments that make these graphs become $C$-UH digraphs are ‘zipped’ into $C$-UH graphs. We do this here for the Pappus graph and the Desargues graph.

Keywords: ultrahomogeneous graph; digraph; shortest cycle; arc-transitivity

1 Preliminaries
The study of ultrahomogeneous graphs (resp. digraphs) can be traced back to [15], [9], [14], [3] and [11], (resp. [8], [13] and [4]). Following a line of research initiated by [12], given a collection $C$ of (di)graphs closed under isomorphisms, a (di)graph $G$ is said to be $C$-ultrahomogeneous (or $C$-UH) if every isomorphism between two induced members of $C$ in $G$ extends to an automorphism of $G$. If $C = \{H\}$ is the isomorphism class of a (di)graph $H$, we say that such a $G$ is $\{H\}$-UH or $H$-UH. In [12], $C$-UH graphs are defined and studied when $C$ is the collection of either (a) the complete graphs, or (b) the disjoint unions of complete graphs, or (c) the complements of those unions.

We may consider a graph $G$ as a digraph by considering each edge $e$ of $G$ as a pair of oppositely oriented (or O-O) arcs $\vec{e}$ and $(\vec{e})^{-1}$. Then, ‘zipping’ $\vec{e}$ and $(\vec{e})^{-1}$ allows to recover $e$, a technique to be used below.

In what follows, new notions for graphs are obtained by deleting “(di)”: for digraphs, by just deleting “(” and “)” around “di”. Let $M$ be a sub(di)graph of a (di)graph $H$ and let $G$ be both an $M$-UH and an $H$-UH (di)graph. We say that a (di)graph $G$ is (fastened) $(H; M)$-UH if, given a copy $H_0$ of $H$ in $G$ containing a copy $M_0$ of $M$, then there exists exactly one copy $H_1 \neq H_0$ of $H$ in $G$ such that $V(H_0) \cap V(H_1) = V(M_0)$ and $A(H_0) \cap \bar{A}(H_1) = A(M_0)$, where $\bar{A}(H_1)$ is formed by those arcs $(\vec{e})^{-1}$ whose orientations are reversed with...
respect to the orientations of the arcs $\tilde{e}$ of $A(H_1)$, and such that no more vertices or arcs than those in $M_0$ are shared by $H_0$ and $H_1$. The directed case here is used in the constructions of Section 3 and in [6, 7]. In the undirected case, the vertex and arc conditions above can be condensed as $H_0 \cap H_1 = M_0$; this is generalized by saying that an $(H; M)$-UH graph $G$ is an $\ell$-fastened $(H; M)$-UH graph if given a copy $H_0$ of $H$ in $G$ containing a copy $M_0$ of $M$, then there exist exactly $\ell$ copies $H_i \neq H_0$ of $H$ in $G$ such that $H_i \cap H_0 = M_0$, for each one of $i = 1, 2, \ldots, \ell$, and such that no more vertices or edges than those in $M_0$ are shared by each two of $H_0, H_1, \ldots, H_\ell$. We work here and in [6, 7] with the cubic distance-transitive (or CDT) graphs $G$, [2]:

<table>
<thead>
<tr>
<th>CDT graph $G$</th>
<th>$a$</th>
<th>$d$</th>
<th>$g$</th>
<th>$k$</th>
<th>$\eta$</th>
<th>$\alpha$</th>
<th>$h$</th>
<th>$h'$</th>
<th>$\kappa$</th>
</tr>
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<tbody>
<tr>
<td>Tetrahedral graph $K_4$</td>
<td>4</td>
<td>1</td>
<td>3</td>
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<td>4</td>
<td>22</td>
<td>0</td>
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<tr>
<td>Thomassen graph $K_{3,3}$</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>9</td>
<td>72</td>
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<td>2</td>
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<tr>
<td>3-cube graph $Q_3$</td>
<td>8</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>6</td>
<td>48</td>
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<tr>
<td>Petersen graph</td>
<td>10</td>
<td>2</td>
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<td>12</td>
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<tr>
<td>Heawood graph</td>
<td>14</td>
<td>3</td>
<td>6</td>
<td>4</td>
<td>28</td>
<td>336</td>
<td>1</td>
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<td>Pappus graph</td>
<td>18</td>
<td>4</td>
<td>7</td>
<td>3</td>
<td>18</td>
<td>216</td>
<td>1</td>
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<td>3</td>
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<td>Dodecahedral graph</td>
<td>20</td>
<td>5</td>
<td>7</td>
<td>3</td>
<td>12</td>
<td>120</td>
<td>0</td>
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<td>Desargues graph</td>
<td>20</td>
<td>5</td>
<td>7</td>
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<td>25</td>
<td>240</td>
<td>1</td>
<td>1</td>
<td>3</td>
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<tr>
<td>Coxeter graph</td>
<td>28</td>
<td>4</td>
<td>7</td>
<td>4</td>
<td>24</td>
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<td>3</td>
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<tr>
<td>Tutte 8-cage</td>
<td>30</td>
<td>4</td>
<td>8</td>
<td>5</td>
<td>90</td>
<td>1440</td>
<td>1</td>
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<td>Foster graph</td>
<td>90</td>
<td>8</td>
<td>10</td>
<td>5</td>
<td>216</td>
<td>4220</td>
<td>1</td>
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<td>Biggs-Smith graph</td>
<td>102</td>
<td>7</td>
<td>9</td>
<td>4</td>
<td>136</td>
<td>2448</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
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where $a, d, g, k, \eta$ and $\alpha$ are order, diameter, girth, AT or arc-transitivity, number of $g$-cycles and number of automorphisms, respectively, with $b$ (resp. $h$) = 1 if $G$ is bipartite (resp. hamiltonian) and = 0 otherwise, and $\kappa$ defined as follows: let $P_k$ and $\tilde{P}_k$ be respectively a $(k-1)$-path and a directed $(k-1)$-path (of length $k-1$); let $C_g$ and $\tilde{C}_g$ be respectively a cycle and a directed cycle of length $g$; then (see Theorem 2 below): $\kappa = 0$, if $G$ is not $(\tilde{C}_g; \tilde{P}_k)$-UH; $\kappa = 1$, if $G$ is planar; $\kappa = 2$, if $G$ is $(\tilde{C}_g; \tilde{P}_k)$-UH with $g < 2(k-1)$; $\kappa = 3$, if $G$ is $(\tilde{C}_g; \tilde{P}_k)$-UH with $g \geq 2(k-1)$.

Given a finite graph $H$ and a subgraph $M$ of $H$ with $|V(H)| > 3$, we say that a graph $G$ is (strongly fastened) SF $(H; M)$-UH if there is a descending sequence of connected subgraphs $M = M_1, \ldots, M_{|V(H)|-2} \equiv K_2$ such that: (a) $M_{i+1}$ is obtained from $M_i$ by the deletion of a vertex, for $i = 1, \ldots, |V(H)| - 3$ and (b) $G$ is a $(2^i - 1)$-fastened $(H; M_i)$-UH graph, for $i = 1, \ldots, |V(H)| - 2$. Theorem 1 below asserts that every CDT graph is an SF $(\tilde{C}_g; \tilde{P}_k)$-UH graph.

Given a graph $C$ and $0 < k \in Z$ such that $k$ at most the diameter of $C$, recall that the $k$-power graph $C_k$ of $C$ has $V(C_k) = V(C)$ and that two vertices are adjacent in $C_k$ if and only if they are at distance $k$ in $C$. Theorem 2 establishes which CDT graphs are $(\tilde{C}_g; \tilde{P}_k)$-UH digraphs. Elevating the resulting oriented cycles to the $(k-1)$-power enables the construction of fastened $C$-UH graphs, with $C$ formed by copies of $K_3$, $K_4$, $C_7$ and $L(Q_3)$, when $\kappa = 3$, via ‘zipping’ of the O-O induced $(k-1)$-arcs shared (as $(k-1)$-paths) by pairs of O-O $g$-cycles. Concretely in Section 3 below, the Pappus (resp. Desargues) graph yields the disjoint union of two copies of the Menger graph of the self-dual $(9_3)$- (resp. $(10_3)$-) configuration, [5]. See also [7].
2  \((C_g, P_k)\)-UH properties of CDT graphs

**Theorem 1** Let \(G\) be a CDT graph of girth \(g\) and \(AT = k\). Then \(G\) is an SF \((C_g; P_{k+2})\)-UH graph, for \(i = 0, 1, \ldots, k - 2\). In particular, \(G\) is a \((C_g; P_k)\)-UH graph and has exactly \(2^{k-2}3ng^{-1}\) \(g\)-cycles.

**Proof.** We have to see that each CDT graph \(G\) with girth \(g\) and \(AT = k\) is a \((2^i - 1)\)-fastened \((C_g; P_{k-i})\)-UH graph, for \(i = 0, 1, \ldots, k - 2\). In fact, each \((k - i - 1)\)-path \(P = P_{k-i}\) of any such \(G\) is shared exactly by \(2^i\) \(g\)-cycles of \(G\), for \(i = 0, 1, \ldots, k - 2\). Moreover, each two of these \(2^i\) \(g\)-cycles have just \(P\) in common. This and a simple counting argument for the number of \(g\)-cycles, as cited in the table above, yield the assertions in the statement.

**Theorem 2** The CDT graphs \(G\) of girth \(g\) and \(AT = k\) that are not \((C_g; P_k)\)-UH digraphs are the Petersen graph, the Heawood graph and the Foster graph. The remaining nine CDT graphs are \((C_g; P_k)\)-UH.

**Proof.** Given a \((C_g; P_k)\)-UH graph \(G\), an assignment of an orientation to each \(g\)-cycle of \(G\) such that the two \(g\)-cycles shared by each \((k - 1)\)-path receive opposite orientations yields a \((C_g; P_k)\)-orientation assignment (or \((C_g; P_k)\)-OA). The collection of \(\eta\) oriented \(g\)-cycles corresponding to the \(\eta\) \(g\)-cycles of \(G\), for a particular \((C_g; P_k)\)-OA, will be called an \((\eta C_g; P_k)\)-OA.

The graph \(G = K_4\) on vertex set \(\{1, 2, 3, 0\}\) admits the \((4 C_4; P_2)\)-OA \(\{(123), (210), (301), (032)\}\). The graph \(G = K_{3,3}\) obtained from \(K_6\) (with vertex set \(\{1, 2, 3, 4, 5, 0\}\)) by deleting the edges of the triangles \(\{1, 3, 5\}\) and \(\{2, 4, 0\}\) admits the \((9 C_4; P_k)\)-OA \(\{(1234), (3210), (4325), (1430), (2145), (0125), (5230), (0345), (5410)\}\). The graph \(G = Q_3\) with vertex set \(\{0, \ldots, 7\}\) and edge set \(\{01, 23, 45, 67, 02, 13, 46, 57, 04, 15, 26, 37\}\) admits the \((6 C_4; P_2)\)-OA \(\{(0132), (1045), (3157), (2376), (0264), (4675)\}\).

If \(G = Pet\) is the Petersen graph, then \(G\) can be obtained from the disjoint union of the \(5\)-cycles \(\mu_\infty = (u_0u_1u_2u_3u_4)\) and \(v_\infty = (v_0v_1v_2v_3v_4)\) by the addition of the edges \((u_x, v_x)\), for \(x \in \mathbb{Z}_5\). Apart from the two \(5\)-cycles given above, the other ten \(5\)-cycles of \(G\) can be denoted by \(\mu_x = (u_{x-1}u_xu_{x+1}v_{x+1}v_x)\) and \(\nu_x = (v_{x-2}v_xv_{x+2}u_{x+2}u_x)\), for each \(x \in \mathbb{Z}_5\). Then the following sequence of alternating \(6\)-cycles and \(2\)-arcs starts and ends with opposite orientations: \(\mu_x^+ (u_3u_2u_1)\mu_x^- (u_0u_1u_2)\mu_x^+ (u_2v_2v_0)\mu_x^- (v_3u_3u_2)\mu_x^+\), where the upper indices \(\pm\) indicate either a forward or backward selection of orientation and each \(2\)-path is presented with the orientation of the previously cited \(5\)-cycle but must be present in the next \(5\)-cycle with its orientation reversed. Thus, \(Pet\) cannot be a \((C_5; P_3)\)-UH digraph.

For each positive integer \(n\), let \(I_n\) stand for the \(n\)-cycle \((0, 1, \ldots, n - 1)\), where \(0, 1, \ldots, n - 1\) are considered as vertices. If \(G = Hea\) is the Heawood graph, then \(G\) can be obtained from \(I_{14}\) by adding the edges \((2x, 5 + 2x)\), for \(x \in \{1, \ldots, 7\}\) where operations are in \(\mathbb{Z}_{14}\). The 28 \(6\)-cycles of \(G\) include the following 7 \(6\)-cycles: \(\gamma_x = (2x, 1+2x, 2+2x, 3+2x, 4+2x, 5+2x)\), where \(x \in \mathbb{Z}_7\).
Then the following sequence of alternating 6-cycles and 3-arcs starts and ends up with opposite orientations for $\gamma_0$:

$$
\gamma_0^+(2345)\gamma_1^-(7654)\gamma_2^+(6789)\gamma_3^-(ba98)\gamma_4^+(abcd)\gamma_5^-(10dc)\gamma_6^+(0123)\gamma_0^-,
$$

(where tridecimal notation is used, up to $d = 13$). Thus, $Hea$ cannot be a $(C_7; P_3)$-UH digraph.

If $G = Pap$ is the Pappus graph, then $G$ can be obtained from $I_{18}$ by adding the edges $(1 + 6x, 6x + 6x), (2 + 6x, 9 + 6x), (4 + 6x, 11 + 6x)$, for $x \in \{0, 1, 2\}$, where operations are mod 18. Then $G$ admits a $(18 C_6; P_3)$-OAC formed by the oriented 6-cycles $A_0 = (123456), B_0 = (3210de), C_0 = (34bced), D_0 = (0165gh), E_0 = (4329ab)$, (where octodecimal notation is used, up to $h = 17$), the 6-cycles $A_x, B_x, C_x, D_x, E_x$ obtained by adding $6x$ mod 18 to the (integer representations of) the vertices of $A_0, B_0, C_0, D_0, E_0$, where $x \in Z_3 \setminus \{0\}$, and finally the 6-cycles $F_0 = (23ef89), F_1 = (hqg5ba), F_2 = (61idc7)$.

If $G = Dod$ is the dodecahedral graph, then $G$ can be seen as a 2-covering graph of the Petersen graph $H$, where each vertex $u_x$, (resp., $v_x$), of $H$ is covered by two vertices $a_x, c_x$, (resp. $b_x, d_x$). This can be done so that a $(12 C_5; P_2)$-OAC of $G$ is formed by the oriented 5-cycles $(a_0a_1a_2a_3a_4), (c_4c_3c_2c_1c_0)$ and for each $x \in Z_5$ also $(a_xd_bx_bx_b2x_2x_1x_1x_1), (d_xb_x+b_2x_2x_2x_2+2x_2x_2)$. If $G = Des$ is the Desargues graph, then $G$ can be obtained from the 20-cycle $I_{20}$, with vertices $4x, 4x + 1, 4x + 2, 4x + 3$ denoted alternatively $x_0, x_1, x_2, x_3$, respectively, for $x \in \{0, \ldots, 4\}$, by adding the edges $(x_3, (x + 2)_0)$, where operations are mod 5. Then $G$ admits a $(20 C_5; P_3)$-OAC formed by the oriented 6-cycles $A_x, B_x, C_x, D_x, E_x, F_x$, for $x \in \{0, \ldots, 4\}$, where

$$
A_x = (x_0x_1x_2x_3(x + 1)_0(x + 4)_1), \quad B_x = (x_0x_1(x + 4)_0(x + 2)_1(x + 2)_2),
$$

$$
C_x = (x_2x_1x_0(x + 3)_2(x + 3)_1x_3), \quad D_x = (x_0(x + 4)_0(x + 1)_1(x + 3)_2x_3).
$$

If $G = Tut$ is Tutte’s 8-cage, then $G$ can be obtained from $I_{30}$, with vertices $5x, 5x + 1, 5x + 2, 5x + 3, 5x + 4, 5x + 5$ denoted alternatively $x_0, x_1, x_2, x_3, x_4, x_5$, respectively, for $x \in Z_5$, by adding the edges $(x_3, (x + 2)_0), (x_1, (x + 1)_4)$ and $(x_2, (x + 2)_3)$. Then $G$ admits the $(90 C_5; P_3)$-OAC formed by the oriented 8-cycles

$$
A^0 = (4_0010_04_00_04_0102_01), \quad B^0 = (4_004_0010_010_02_02_02_0), \quad C^0 = (0_004_004_0010_04_02_02_0), \quad F^0 = (4_004_004_0010_04_02_02_0),
$$

$$
F^0 = (0_004_004_0010_04_02_02_0), \quad L^0 = (0_004_0010_04_02_02_02_0), \quad O^0 = (3_03_03_32_03_03_03_0), \quad R^0 = (3_03_03_32_03_03_03_0),
$$

$$
R^0 = (0_004_004_0010_04_02_02_0). \quad P^0 = (4_004_004_0010_04_02_02_0), \quad Q^0 = (4_004_004_0010_04_02_02_0), \quad R^0 = (0_004_004_0010_04_02_02_0),
$$

together with those obtained from these 18 8-cycles by adding $x \in Z_5$ uniformly mod 5 to all subindices. Accordingly, these 8-cycles are denoted $A^x, \ldots, R^x$, where $x \in Z_5$.

If $G = Fos$ is the Foster graph, then $G$ can be obtained from $I_{90}$, with vertices $5x, 5x + 1, 5x + 2, 5x + 3, 5x + 4, 5x + 5$ denoted alternatively $x_0, x_1, x_2, x_3, x_4, x_5$, respectively, for $x \in Z_5$, by adding the edges $(x_4, (x + 2)_1), (x_0, (x + 2)_5)$. The 90 10-cycles of $G$ include the following 15 10-cycles, where $x \in Z_{15}$.
\( \phi_x = (x_2 x_3 (x+1) a(x+1) (x+2) (x+1) (x+1) a(x+1) (x+2) a(x+2) ) \).

Then the following sequence of alternating 10-cycles and 4-arcs:

\[
\phi_x^{[14]} \phi_x^{[3]} \phi_x^{[3]} \phi_x^{[5]} \phi_x^{[5]} \phi_x^{[5]} \phi_x^{[7]} \phi_x^{[7]} \phi_x^{[9]} \phi_x^{[9]} \phi_x^{[9]} \phi_x^{[9]},
\]

may be continued with \( \phi_0^{\circ} \), of opposite orientation to that of the initial \( \phi_0^\circ \),
where \([x_j]\) stands for a 3-path starting at the vertex \( x_j \) in the previously cited (to the left) oriented 10-cycle. Thus, \( \text{Pos} \) cannot be a \((\bar{C}_{10}; \bar{P}_5)\)-UH digraph.

The cases of the Coxeter and Biggs-Smith graphs are treated respectively in the Theorem 2 of [6] and the Theorem 2 of [7].

\[ \square \]

3 ‘Zipping’ the \((k - 1)\)-powers of \(g\)-cycles

Given a CDT graph \( G \) with \( \kappa = 3 \), consider the collection \( C_{g}^{k-1}(G) \) of \((k - 1)\)-powers of oriented \( g \)-cycles in the \((\eta \bar{C}_g; \bar{P}_k)\)-OAC of \( G \) in the proof of Theorem 2. If \( k = 3 \), then each arc \( \bar{e} \) of a member \( C_3^2(G) \) is marked by the middle vertex of the 2-arc \( \bar{E} \) in \( C \) for which \( \bar{e} \) stands, while the tail and head of \( \bar{e} \) are marked by the tail and head of \( \bar{E} \), respectively. This is the case below: we consider the CDT graphs \( G \) with \( \kappa = k = 3 \) in order to ‘zip’ such \( C_3^2 \) along their O-O arc pairs to obtain corresponding graphs \( Y(G) \) with C-UH properties. In all these cases, the following sequence of operations is performed:

\[
G \rightarrow (\eta \bar{C}_g; \bar{P}_k)\text{-OAC}(G) \rightarrow C_{g}^{k-1}(G) \rightarrow Y(G).
\]

The CDT graphs \( G \) with \( \kappa = 0 \) do not admit the approach suggested in the previous paragraph for their \( g \)-cycles lack a \((\bar{C}_g; \bar{P}_k)\)-OA; those with \( \kappa = 1 \) admit the approach with \( Y(G) = G \) so nothing new is obtained more than a corresponding polyhedral graph (embeddable into the sphere) with faces delimited by \( g \)-cycles, namely the tetrahedral, 3-cube and dodecahedral graphs; those with \( \kappa = 2 \) again admit the approach, but since \( \kappa/2 = k - 1 \), then \( Y(G) = (g - 1)C_{k-1} \), the multigraph of multiplicity \( g - 1 \) on the \((k - 1)\)-th power of \( G \).

If \( G \) is either the Pappus graph \( \text{Pap} \) or the Desargues graph \( \text{Des} \), then \( C_{3}^{2}(G) \) is formed by triangles conforming a graph \( Y(G) \) with just two connected components \( Y_1(G) \) and \( Y_2(G) \).

Each of \( Y_1(\text{Pap}) \) and \( Y_2(\text{Pap}) \) is embeddable in a closed orientable surface \( T \) of genus 1, or 1-torus. In fact, Figure 1 shows toroidal cutouts of \( Y_1(\text{Pap}) \) and \( Y_2(\text{Pap}) \). Notice that the copies of \( K_3 \) in \( C_{3}^{2}(G) \) are contractible in \( T \). These triangles form two collections \( \mathcal{H}_1, \mathcal{H}_2 \) of copies \( y_j^{i} \) of \( K_3 \) closed under parallel translation, where \( y = A, B, C, D, E, F; i = 0, 1, 2 \) and \( j = 1, 2 \), namely: the nine of \( \mathcal{H}_1(\mathcal{H}_2) \) with horizontal edge below (above) its opposite vertex. There is also a collection \( \mathcal{H}_0 \) of nine non-contractible copies of \( K_3 \) in \( G \), traceable
linearly in three different parallel directions, three such triangles per direction, with the edges of each triangle marked by an associated common vertex of $Pap$.

Theorem 3: $Y_1(Pap)$ and $Y_2(Pap)$ are isomorphic $K_2$-fastened $\{H_0, H_1, H_2\}$-UH graphs, where $H_i$ is a representative of $H_i$, for $i = 0, 1, 2$. Moreover, each of $Y_1(Pap)$ and $Y_2(Pap)$ can be taken as the Menger graph of the Pappus self-dual $(9_3)$-configuration in 12 different forms, by selecting the point set $P$ and the line set $L \neq P$ so that $\{P, L\} \subset \{V(Pap), H_0, H_1, H_2\}$ and the incidence relation either as the inclusion of a vertex in a copy of $K_3$ or as the containment by a copy of $K_3$ of a vertex or as the sharing of an edge by two copies of $K_3$.

Proof. The statement can be established by managing the data given above. The 12 different claimed forms correspond to the arcs of the complete graph on vertex set $\{V(Pap), H_0, H_1, H_2\}$. □

If $G = Des$, then $Y_1(G)$ and $Y_2(G)$ are isomorphic $K_2$-fastened $(K_4, K_3)$-UH graphs, each formed by five copies of $K_4$ and ten copies of $K_3$, with each such copy of $K_3$: (a) not forming part of a copy of $K_4$ in $Y_1(G)$ or $Y_2(G)$; (b) having its edges marked by a constant symbol, as shown in Figure 2.

Deleting a copy $H$ of $K_4$ from such $Y_i(Des)$ yields a copy of $K_{2,2,2}$, four of whose composing copies of $K_3$, with no common edges, are faces of corresponding copies of $K_4 \neq H$; the other four copies of $K_3$ are among the ten mentioned copies of $K_3$ in $G$. A realization of $Y_i(G)$ (or $Y_2(G)$) in 3-space can be obtained from a regular octahedron $O_3$ realizing the $K_{2,2,2}$ cited above via the midpoints of the four segments joining the barycenters of four edge-disjoint

\[\text{Figure 1: Toroidal cutouts of } Y_1(Pap) \text{ and } Y_2(Pap)\]
alternate triangles in $O_3$ to the barycenter of $O_3$ by constructing the tetrahedra determined by each of these alternate triangles and the nearest constructed midpoint, as well as the fifth central tetrahedron determined by the four midpoints.

By considering the barycenters of the resulting five tetrahedra and the segments joining them, a copy of $K_5$ in 3-space is obtained. The geometric line graph $L(K_5)$ it gives place to appears as a smaller version of $Y_1(G)$ (or $Y_2(G)$) contained in an octahedron $O_3 \subset O_3$. This procedure may be repeated indefinitely, generating a sequence of realizations of $Y_1(G)$ (or $Y_2(G)$) in 3-space. Since $Y_1(Des)$ and $Y_2(Des)$ are isomorphic to $L(K_5)$, whose complement is Pet, then this sequence yields a corresponding sequence of realizations of Pet in 3-space.

We notice that the ten vertices and ten copies of $K_3$ of either $Y_i(Des)$ ($i = 1, 2$) may be considered as the points and lines of the Desargues self-dual ($10_3$) configuration, and that the Menger graph of this coincides with $Y_i(Des)$, [5]. Each vertex of $Y_i(Des)$ is the meeting vertex of two copies of $K_4$ and three copies of $K_3$ not forming part of a copy of $K_4$.

**Theorem 4** $Y_1(Des)$ and $Y_2(Des)$ are $K_2$-fastened $\{K_4, K_3\}$-UH graphs composed by five copies of $K_4$ and ten copies of $K_3$ each. Moreover, the ten vertices and ten copies of $K_3$ in either graph constitute the Desargues self-dual ($10_3$) configuration, which has the graph itself as its Menger graph. Furthermore, both graphs are isomorphic to $L(K_5)$, whose complement is the Petersen graph. □

Theorem 4 can be partly generalized by replacing $L(K_5)$ by $L(K_n)$ ($n \geq 4$). This produces a $K_2$-fastened $\{K_{n-1}, K_3\}$-UH graph.

**Theorem 5** The line graph $L(K_n)$, with $n \geq 4$, is a $K_2$-fastened $\{K_{n-1}, K_3\}$-UH graph with $n$ copies of $K_{n-1}$ and \( \binom{n}{3} \) copies of $K_3$.

**Proof.** We assume that each vertex of $K_n$ is taken as a color of edges of $L(K_n)$ under the following rule: Color all the edges between vertices of $L(K_n)$ representing edges incident to a vertex $v$ of $K_n$ with color $v$. Then, each triple of
edge colors for \(L(K_n)\) corresponds to the edges of a well determined copy of \(K_3\) in \(L(K_n)\). Thus, there are exactly \(\binom{n}{3}\) copies of \(K_3\) intervening in \(L(K_n)\) looked upon as a \(\{K_{n-1}, K_3\}\)-UH graph. \qed

References


[6] I. J. Dejter, From the Coxeter graph to the Klein graph, manuscript 2010.


