On a $\vec{C}_4$-ultrahomogeneous oriented graph

Italo J. Dejter
University of Puerto Rico
Rio Piedras, PR 00931-3355
idejter@uprrp.edu

Abstract

The notion of a $\mathcal{C}$-ultrahomogeneous graph, due to Isaksen et al., is adapted for digraphs, and then a strongly connected $\vec{C}_4$-ultrahomogeneous oriented graph on 168 vertices and 126 pairwise arc-disjoint 4-cycles is presented, with regular indegree and outdegree 3 and no circuits of lengths 2 and 3, by altering a definition of the Coxeter graph via pencils of ordered lines of the Fano plane in which pencils are replaced by ordered pencils.

Keywords: ultrahomogeneous oriented graph; Fano plane; ordered pencil

1 Introduction

The study of ultrahomogeneous graphs (resp. digraphs) can be traced back to [12], [6], [11] and, [7], (resp. [5], [10] and [2]). In [9], $\mathcal{C}$-ultrahomogeneous graphs are defined and subsequently treated when $\mathcal{C}$ = collection of either **(a)** complete graphs, or **(b)** disjoint unions of complete graphs, or **(c)** complements of those unions. In [3], a $\{K_4, K_{2,2,2}\}$-ultrahomogeneous graph on 42 vertices, 42 copies of K_4 and 21 copies of $K_{2,2,2}$ is given that fastens objects of (a) and (c), namely K_4 and $K_{2,2,2}$, respectively, over copies of K_2.

In the present note and in [4], the notion of a $\mathcal{C}$-ultrahomogeneous graph is extended as follows: Given a collection $\mathcal{C}$ of (di)graphs closed under isomorphisms, a (di)graph G is $\mathcal{C}$-*ultrahomogeneous* (or $\mathcal{C}$-UH) if every isomorphism between two G-induced members of $\mathcal{C}$ extends to an automorphism of G. If $\mathcal{C} = \{H\}$ is the isomorphism class of a (di)graph H, such a G is said to be $\{H\}$-UH or H-UH.

In [4], the cubic distance-transitive graphs are shown to be C_g-UH graphs, where C_g stands for cycle of minimum length, i.e. realizing the girth g; moreover, all these graphs but for the Petersen, Heawood and Foster graphs are shown to be $\vec{C}_g$-UH digraphs, which allows the construction of novel $\mathcal{C}$-UH graphs, in continuation to the work of [3], including a $\{K_4, L(Q_3)\}$-UH graph on 102 vertices that fastens 102 copies of K_4 and 102 copies of the cuboctahedral graph $L(Q_3)$ over copies of K_3, obtained from the Biggs-Smith graph, ([1]), by unzipping, powering and zipping back a collection of oriented g-cycles provided by the initial results. However, these graphs are undirected, so they are not properly digraphs.

In this note, a presentation of the Coxeter graph Cox is modified to provide a strongly connected $\vec{C}_4$-UH oriented graph D on 168 vertices, 126 pairwise arc-disjoint 4-cycles, with regular indegree and outdegree 3. In contrast, the construction of [3] used ordered pencils of unordered lines, instead.

We take the Fano plane $\mathcal{F}$ as having point set $J_7 = \mathbf{Z}_7$ (the cyclic group mod 7) and point-line correspondence $\phi(j) = \{(j+1), (j+2), (j+4)\}$, for every $j \in \mathbf{Z}_7$ in order to color the vertices and edges of Cox as in Figure 1.

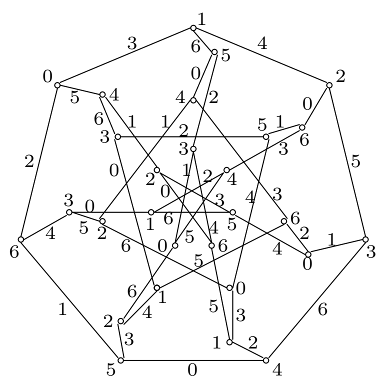

Figure 1: Coloring the vertices and edges of Cox with elements of $\mathcal{F}$

This figure shows that each vertex v of Cox can be considered as an unordered pencil of ordered lines of $\mathcal{F}$, (brackets and commas avoided now):

$$xb_1c_1, \ \ xb_2c_2, \ \ xb_0c_0, \tag{1}$$

corresponding to the three edges e_1, e_2, e_0 incident to v, respectively, and denoted by $[x, b_1c_1, b_2c_2, b_0c_0]$, where x is the color of v in the figure, with b_i and c_i as the colors of the edge e_i and the endvertex of e_i other than v, for $i \in \{1, 2, 0\}$.

Moreover, two such vertices are adjacent in Cox if they can be written $[x, b_1c_1, b_2c_2, b_0c_0]$ and $[x', b'_1c'_1, b'_2c'_2, b'_0c'_0]$ (perhaps by means of a permutation of the entries b_ic_i) in such a way that $\{b_i, c_i\} \cap \{b'_i, c'_i\}$ is constituted by just one element d_i, for each $i \in \{1, 2, 0\}$, and the resulting triple $d_1d_2d_0$ is a line of $\mathcal{F}$.

2 Presentation of a $\vec{C}_4$-UH digraph

Consider the oriented graph D whose vertices are the *ordered* pencils of ordered lines of $\mathcal{F}$, as in (1) above. Each such vertex will be denoted $(x, b_1c_1, b_2c_2, b_0c_0)$, where $b_1b_2b_0$ is a line of $\mathcal{F}$. An arc between two vertices of D, say from $(x, b_1c_1, b_2c_2, b_0c_0)$ and $(x', b'_1c'_1, b'_2c'_2, b'_0c'_0)$, is established if and only if

$$\begin{array}{llll} x = c'_i, & b'_{i+1} = c_{i+1}, & b'_{i-1} = c_{i-1}, & b'_i = b_i, \\ x' = c_i, & c'_{i+1} = b_{i-1}, & c'_{i-1} = b_{i+1}, & \end{array}$$

for some, $i \in \{1,2,0\}$. This way, we obtain oriented 4-cycles in D, such as

$$((0,26,54,31),(6,20,43,15),(0,26,31,54),(6,20,15,43)).$$

A simplified notation for the vertices (x, yz, uv, pq) of D is yup_x. With such a notation, the adjacency sub-list of D departing from the vertices of the form yup_0 is (with rows indicated a, b, c, d, e, f, to be used below):

$124_0 : 165_3, 325_6, 364_5;$	$235_0 : 214_6, 634_1, 615_6;$	$346_0 : 352_1, 142_5, 156_2;$	$156_0 : 142_3, 352_4, 346_2;$
$142_0 : 156_3, 346_5, 352_6;$	$253_0 : 241_6, 651_4, 643_6;$	$364_0 : 325_1, 165_2, 124_5;$	$165_0 : 124_3, 364_2, 325_4;$
$214_0 : 235_6, 615_3, 634_5;$	$325_0 : 364_1, 124_6, 165_1;$	$436_0 : 412_5, 532_1, 516_2;$	$516_0 : 532_4, 412_3, 436_2;$
$241_0 : 253_6, 643_5, 651_3;$	$352_0 : 346_1, 156_4, 142_1;$	$463_0 : 421_5, 561_2, 523_1;$	$561_0 : 523_4, 463_2, 421_3;$
$412_0 : 436_5, 516_3, 532_6;$	$523_0 : 561_4, 421_6, 463_4;$	$634_0 : 615_2, 235_1, 214_5;$	$615_0 : 634_2, 214_3, 235_4;$
$421_0 : 463_5, 523_6, 561_3;$	$532_0 : 516_4, 436_1, 412_4;$	$643_0 : 651_2, 241_5, 253_1;$	$651_0 : 643_2, 253_4, 241_3.$

From this sub-list, the adjacency list of D, for its $168 = 24 \times 7$ vertices, is obtained via translations mod 7. Let us represent each vertex yup_0 of D by means of a symbol j_i, where $i = a, b, c, d, e, f$ stands for the successive rows of the table above and $j = \phi^{-1}(yup) \in \{0,1,2,4\}$. These symbols j_i are assigned to the lines yup avoiding $0 \in \mathcal{F}$, and thus to the vertices yup_0, as follows:

j_i	j=0	j=1	j=2	j=4
i=a	124	235	346	156
i=b	142	253	364	165
i=c	214	325	436	516
i=d	241	352	463	561
i=e	412	523	634	615
i=f	421	532	643	651

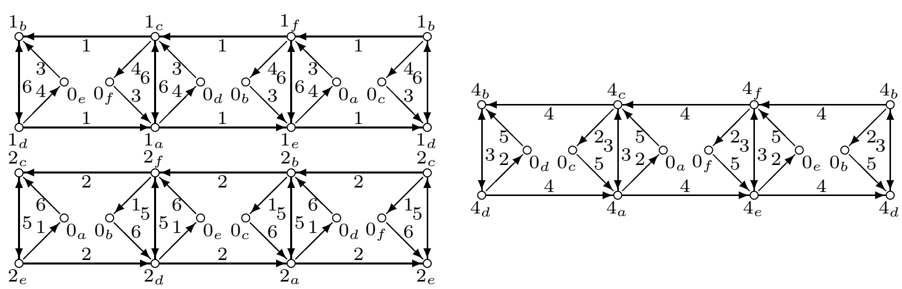

Figure 2: Split representation of the quotient graph $D/\mathbf{Z}_7$

With these symbols adopted, the quotient graph $D/\mathbf{Z}_7$ can be considered as a voltage graph with group $\mathbf{Z}_7$ and derived graph D, ([8]), admitting a split representation into the three connected digraphs of Figure 2, whose vertices are indicated by the symbols j_i of their representatives yup_0, and in which: **(1)** the 18 oriented 4-cycles that are shown are interpreted all with counterclockwise orientation; **(2)** For each $i \in \{a, \dots, f\}$, the three vertices indicated by 0_i represent just one vertex of D/Z_7, so they must be identified; **(3)** the leftmost arc in each one of the three connected digraphs must be identified with the corresponding rightmost arc by parallel translation; **(4)** if an arc $\vec{e}$ of $D/\mathbf{Z}_7$ has voltage $\nu \in \mathbf{Z}_7$, initial vertex j_i and terminal vertex $j'_{i'}$, then a representative

yup_μ of j_i initiates an arc in D that covers $\vec{e}$ and has terminal vertex $y'u'p'_{(\nu+\mu)}$, where yup_0 and $y'u'p'_0$ are represented respectively by j_i and $j'_{i'}$.

All the oriented 4-cycles of D are obtained by uniform translations mod 7 from these 18 oriented 4-cycles. Thus, there are just $126 = 7 \times 18$ oriented 4-cycles of D. Our construction of D shows that the following statement holds.

Theorem 1 *The oriented graph D is a strongly connected $\vec{C}_4$-UH digraph on* 168 *vertices,* 126 *pairwise disjoint oriented* 4*-cycles, with regular indegree and outdegree both equal to* 3 *and no circuits of lengths* 2 *and* 3*.* □

References

[1] N. L. Biggs and D. H. Smith, *On trivalent graphs*, Bull. London Math. Soc., **3**(1971), 155-158.

[2] G. L. Cherlin, The Classification of Countable Homogeneous Directed Graphs and Countable Homogeneous n-tournaments, Memoirs Amer. Math. Soc., vol. 131, number 612, Providence RI, January 1988.

[3] I. J. Dejter, *On a $\{K_4, K_{2,2,2}\}$-ultrahomogeneous graph*, Australasian Journal of Combinatorics, **44**(2009), 63–76.

[4] I. J. Dejter, *On certain $\mathcal{C}$-ultrahomogeneous graphs obtained from cubic distance-transitive graphs*, preprint, 2009.

[5] R. Fraïssé, *Sur l'extension aux relations de quelques proprietés des ordres*, Ann. Sci. École Norm. Sup. 71 (1954), 363–388.

[6] A. Gardiner, *Homogeneous graphs*, J. Combinatorial Theory (B), **20** (1976), 94–102.

[7] Ja. Ju. Gol'fand and M. H. Klin, *On k-homogeneous graphs*, (Russian); Algorithmic studies in combinatorics (Russian), pp. 76–85, 186 (errata insert), "Nauka", Moscow, 1978.

[8] J. Gross and T. W. Tucker, Topological Graph Theory, New York: Wiley, 1987.

[9] D. C. Isaksen, C. Jankowski and S. Proctor, *On K_*-ultrahomogeneous graphs*, Ars Combinatoria, Volume LXXXII, (2007), 83–96.

[10] A. H. Lachlan and R. Woodrow, *Countable ultrahomogeneous undirected graphs*, Trans. Amer. Math. Soc. 262 (1980), 51-94.

[11] C. Ronse, *On homogeneous graphs*, J. London Math. Soc. (2) **17** (1978), 375–379.

[12] J. Sheehan, *Smoothly embeddable subgraphs*, J. London Math. Soc. (2) **9** (1974), 212–218.