

On a \vec{C}_4 -ultrahomogeneous oriented graph

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Abstract

The notion of a \mathcal{C} -ultrahomogeneous graph, due to Isaksen et al., is adapted for digraphs, and then a strongly connected \vec{C}_4 -ultrahomogeneous oriented graph on 168 vertices and 126 pairwise arc-disjoint 4-cycles is presented, with regular indegree and outdegree 3 and no circuits of lengths 2 and 3, by altering a definition of the Coxeter graph via pencils of ordered lines of the Fano plane in which pencils are replaced by ordered pencils.

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1 Introduction

The study of ultrahomogeneous graphs (resp. digraphs) can be traced back to [12], [6], [11] and, [7], (resp. [5], [10] and [2]). In [9], \mathcal{C} -ultrahomogeneous graphs are defined and subsequently treated when \mathcal{C} = collection of either (a) complete graphs, or (b) disjoint unions of complete graphs, or (c) complements of those unions. In [3], a $\{K_4, K_{2,2,2}\}$ -UH on 42 vertices, 42 copies of K_4 and 21 copies of $K_{2,2,2}$ is given that fastens objects of (a) and (c), namely K_4 and $K_{2,2,2}$, respectively, over copies of K_2 .

In the present note and in [4], the notion of a \mathcal{C} -ultrahomogeneous graph is extended as follows: Given a collection \mathcal{C} of (di)graphs closed under isomorphisms, a (di)graph G is \mathcal{C} -ultrahomogeneous (or \mathcal{C} -UH) if every isomorphism between two G -induced members of \mathcal{C} extends to an automorphism of G . If $\mathcal{C} = \{H\}$ is the isomorphism class of a (di)graph H , such a G is said to be $\{H\}$ -UH or H -UH.

In [4], the cubic distance-transitive graphs are shown to be C_g -UH graphs, where C_g stands for cycle of minimum length, i.e. realizing the girth g ; moreover, all these graphs but for the Petersen, Heawood and Foster graphs are shown to be \vec{C}_g -UH digraphs, which allows the construction of novel \mathcal{C} -UH graphs, in continuation to the work of [3], including a $\{K_4, L(Q_3)\}$ -UH graph on 102 vertices that fastens 102 copies of K_4 and 102 copies of the cuboctahedral graph $L(Q_3)$ over copies of K_3 , obtained from the Biggs-Smith graph by unzipping,

powering and zipping back a collection of oriented g -cycles provided by the initial results. However, these graphs are undirected, so they are not properly digraphs.

In this note, a presentation of the Coxeter graph Cox is modified to provide a strongly connected \vec{C}_4 -UH oriented graph D on 168 vertices, 126 pairwise arc-disjoint 4-cycles, with regular indegree and outdegree 3. In contrast, the construction of [3] used ordered pencils of unordered lines, instead.

We take the Fano plane \mathcal{F} as having point set $J_7 = \{0, 1, \dots, 6\}$ and line set $\{124, 235, 346, 450, 561, 602, 013\}$, in order to color the vertices and edges of Cox as in Figure 1.

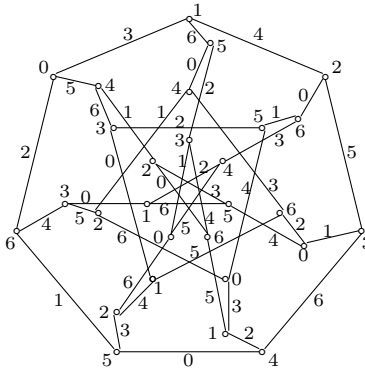


Figure 1: Coloring the vertices and edges of Cox with elements of \mathcal{F}

This figure suggest that each vertex v of Cox can be considered as an unordered pencil of ordered lines of \mathcal{F} :

$$xb_1c_1, xb_2c_2, xb_0c_0, \quad (1)$$

corresponding to the three edges e_1, e_2, e_0 incident to v , respectively, and denoted by $[x, b_1c_1, b_2c_2, b_0c_0]$, where x is the color of v in the figure, with b_i and c_i as the colors of the edge e_i and the endvertex of e_i other than v , for $i \in \{1, 2, 0\}$.

Moreover, two such vertices are adjacent in Cox if they can be written $[x, b_1c_1, b_2c_2, b_0c_0]$ and $[x', b'_1c'_1, b'_2c'_2, b'_0c'_0]$ in such a way that $\{b_i, c_i\} \cap \{b'_i, c'_i\}$ is constituted by just one element d_i , for each $i \in \{1, 2, 0\}$, and the resulting triple $d_1d_2d_0$ is a line of \mathcal{F} .

2 Presentation of a \vec{C}_4 -UH digraph

Consider the oriented graph D whose vertices are the *ordered* pencils of ordered lines of \mathcal{F} , as in (1) above. Each such vertex will be denoted $(x, b_1c_1, b_2c_2, b_0c_0)$, where $b_1b_2b_0$ is a line of \mathcal{F} . An arc between two vertices of D , say from $(x, b_1c_1, b_2c_2, b_0c_0)$ and $(x', b'_1c'_1, b'_2c'_2, b'_0c'_0)$, is established if and only if

$$\begin{aligned} x &= c'_i, & b'_{i+1} &= c_{i+1}, & b'_{i-1} &= c_{i-1}, & b'_i &= b_i, \\ x' &= c_i, & c'_{i+1} &= b_{i-1}, & c'_{i-1} &= b_{i+1}, \end{aligned}$$

for some, $i \in \{1, 2, 0\}$. This way, we obtain oriented 4-cycles in D , such as

$$((0, 26, 54, 31), (6, 20, 43, 15), (0, 26, 31, 54), (6, 20, 15, 43)).$$

A simplified notation for the vertices (x, yz, uv, pq) of D is yup_x . With such a notation, the adjacency sub-list of D departing from the vertices of the form yup_0 is (with rows indicated a, b, c, d, e, f , to be used below):

124 ₀ : 165 ₃ , 325 ₆ , 364 ₅ ;	235 ₀ : 214 ₆ , 634 ₁ , 615 ₆ ;	346 ₀ : 352 ₁ , 142 ₅ , 156 ₂ ;	156 ₀ : 142 ₃ , 352 ₄ , 346 ₂ ;
142 ₀ : 156 ₃ , 346 ₅ , 352 ₆ ;	253 ₀ : 241 ₆ , 651 ₄ , 643 ₆ ;	364 ₀ : 325 ₁ , 165 ₂ , 124 ₅ ;	165 ₀ : 124 ₃ , 364 ₂ , 325 ₄ ;
214 ₀ : 235 ₆ , 615 ₃ , 634 ₅ ;	325 ₀ : 364 ₁ , 124 ₆ , 165 ₁ ;	436 ₀ : 412 ₅ , 532 ₁ , 516 ₂ ;	516 ₀ : 532 ₄ , 412 ₃ , 436 ₂ ;
241 ₀ : 253 ₆ , 643 ₅ , 651 ₃ ;	352 ₀ : 346 ₁ , 156 ₄ , 142 ₁ ;	463 ₀ : 421 ₅ , 561 ₂ , 523 ₁ ;	561 ₀ : 523 ₄ , 463 ₂ , 421 ₃ ;
412 ₀ : 436 ₅ , 516 ₃ , 532 ₆ ;	523 ₀ : 561 ₄ , 421 ₆ , 463 ₄ ;	634 ₀ : 615 ₂ , 235 ₁ , 214 ₅ ;	615 ₀ : 634 ₂ , 214 ₃ , 235 ₄ ;
421 ₀ : 463 ₅ , 523 ₆ , 561 ₃ ;	532 ₀ : 516 ₄ , 436 ₁ , 412 ₄ ;	643 ₀ : 651 ₂ , 241 ₅ , 253 ₁ ;	651 ₀ : 643 ₂ , 253 ₄ , 241 ₃ ;

From this sub-list, the adjacency list of D , for its $168 = 24 \times 7$ vertices, is obtained via translations mod 7. Let us represent each vertex yup_0 of D by means of a symbol i_j , where $j = a, b, c, d, e, f$ stands for the successive rows of the table above and $i \in \{0, 1, 2, 4\}$. These symbols i_j are assigned to the lines yup avoiding $0 \in \mathcal{F}$, and thus to the yup_0 , as follows:

i_j	$j=a$	$j=b$	$j=c$	$j=d$	$j=e$	$j=f$
$i=0$	124 ₀	142 ₀	214 ₀	241 ₀	412 ₀	421 ₀
$i=1$	235 ₀	253 ₀	325 ₀	352 ₀	523 ₀	532 ₀
$i=2$	346 ₀	364 ₀	436 ₀	463 ₀	634 ₀	643 ₀
$i=4$	156 ₀	165 ₀	516 ₀	561 ₀	615 ₀	651 ₀

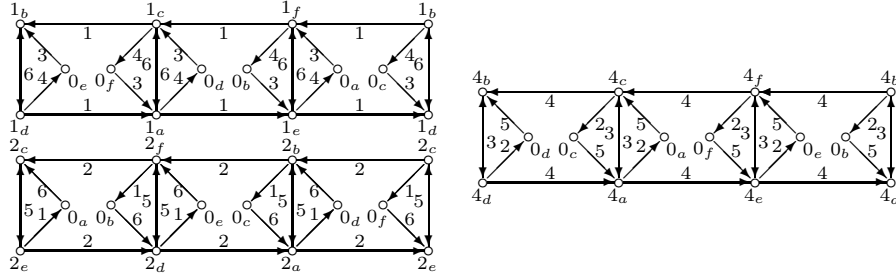


Figure 2: Split representation of the quotient graph D/\mathbf{Z}_7

With these symbols adopted, the quotient graph D/\mathbf{Z}_7 can be considered as a voltage graph with group \mathbf{Z}_7 and derived graph D , ([8]), admitting a split representation into the three connected digraphs of Figure 2, whose vertices are indicated by the symbols i_j of their representatives yup_0 , and in which: **(1)** the 18 oriented 4-cycles that are shown are interpreted all with counterclockwise orientation; **(2)** the three vertices indicated by 0_j , for each $j \in \{a, \dots, f\}$, represent just one vertex of D/\mathbf{Z}_7 , so they must be identified; **(3)** the leftmost arc in each one of the three connected digraphs must be identified with the corresponding rightmost arc by parallel translation; **(4)** if an arc \vec{e} of D/\mathbf{Z}_7 has voltage $\nu \in \mathbf{Z}_7$, initial vertex i_j and terminal vertex i'_j , then a representative yup_μ of i_j initiates an arc in D that covers \vec{e} and has terminal vertex $y'u'p'_{\nu+\mu}$, where yup_0 and $y'u'p'_0$ are represented by i_j and i'_j .

All the oriented 4-cycles of D are obtained by uniform translations mod 7 from these 18 oriented 4-cycles. Thus, there are just $126 = 7 \times 18$ oriented 4-cycles of D . Our construction of D shows that the following statement holds.

Theorem 1 *The oriented graph D is a strongly connected \vec{C}_4 -UH digraph on 168 vertices, 126 pairwise disjoint oriented 4-cycles, with regular indegree and outdegree both equal to 3 and no circuits of lengths 2 and 3.* \square

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