On Unitary Cayley Graphs

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ABSTRACT. We deal with a family of undirected Cayley graphs $X_*$ which are unions of disjoint Hamilton cycles, and some of their properties, where $n$ runs over the positive integers. It is proved that $X_*$ is a bipartite graph when $n$ is even. If $n$ is an odd number, we count the number of different colored triangles in $X_*$. 

1. Introduction

The graphs considered in this paper are undirected, simple and without loops. Given a graph $X_n$, we denote its vertex set and edge set by $V(X)$ and $E(X)$, respectively. Given a positive integer $n$ and an element $x$ of the additive cyclic group $Z_n$, of integer residue classes mod $n$, there is only one integer representative $y$ of $x$ such that $0 < y < n$; we use the same symbol $x$ to denote $y$ and define $|x| = x$ if $x \leq n/2$ and $|x| = n - x$ if $x > n/2$.

The group $Z_n$ possesses the subset $U_n = U(Z_n)$ of units modulo $n$, constituted by those $x$ in $Z_n$ with integer representatives relatively prime to $n$. It is known that $U_n$ is a multiplicative group. Let $W_n = U_n/Z_2$ be given by the classes $\{x, -x\}$, where $x$ varies in $U_n$. Notice that $W_n$ inherits a group structure from $U_n$. We represent each $\{x, -x\} \in W_n$ again by $x$, where $0 < x \leq \lfloor n/2 \rfloor$, unless confusion arises.

We deal with a class $\{X_n\}$ of undirected Cayley graphs $X_n$, that we call "Unitary Cayley graphs", defined as follows: $V(X_n) = Z_n$ and any two vertices $v_1$ and $v_2$ are adjacent if and only if $|v_1 - v_2| \in U_n$. Each edge
of \( X_n \) is attributed a color from the set \( \{1, 2, \ldots, k\} \), where \( k = [n/2] \) according to the following rule: If \( e \) is an edge with endvertices \( v_1 \) and \( v_2 \) and if \( v_1 - v_2 \equiv \pm i \pmod{n} \), \( i \in \{1, 2, \ldots, k\} \), then we color the edge \( e \) with \( i \). According to the definition of \( X_n \), we have that \( i \in U_n \). Therefore, \( X_n \) is the undirected Cayley graph of \( Z_n \) with generator set \( W_n \). Written Cay \([Z_n, W_n]\), hence, we may say that \( X_n \) has an edge coloring with one half of the unit element of \( Z_n \). For instance, if \( n = 9 \) then \( U_n = \{1, 2, 4, 5, 7, 8\} \) and \( W_n = \{1, 2, 4\} \). Thus, \( X_9 = \text{Cay} [Z_9, W_9] \) is represented in the Figure 1.1.

In [1] Dejter dealt with Cayley graphs of the form Cay \([Z_n, I_n]\), where \( I_n \) is the generator set \( \{1, 2, \ldots, k\} \), \( n \) is odd and \( k = (n - 1)/2 \), i.e., the complete graphs \( K_n \) edge-colored in a symmetric fashion, and studied some induced subgraphs \( K_r \) of these Cayley graphs that were called totally multicolored (TMC) subgraphs, motivated by a question of Erdös, Pyber and Tuza [2].

![Figure 1.1]

In the present paper, we deal with some properties of the \( X_n \). In particular, besides proving that if \( n \) is even, then \( X_n \) is bipartite we count the number of different colored triangles of \( X_n \), when \( n \) is odd.

2. Basic Properties of Unitary Cayley Graphs

We now state some properties of the Unitary Cayley graphs. The proofs are easy and left to the reader.

**Proposition 2.1.** \( X_n \) is a regular graph of degree \( \phi(n) \), where \( \phi(n) \) is the Euler \( \phi \)-function, and is a union of \( \phi(n) \) Hamilton cycles of length \( n \), for any positive integer \( n \).

**Proposition 2.2.** \( X_n \) is isomorphic to a complete graph with \( n \) vertices, if \( n \) is a prime number and isomorphic to a complete bipartite graph \( K(2^{t-1}, 2^{t-1}) \) if \( n = 2^t \).
The graphs $X_n$ for $n = 2^t$ are interesting from the chromatic point of view. They are regular of degree one half of the number of vertices and the number of edges turns out to be the square of the degree of the graph. These graphs are a special type studied in [3]; they are chromatically unique.

Unitary Cayley graphs have different properties according to the parity of $n$, as will be seen in the next proposition.

**Proposition 2.3.** $X_n$ is a bipartite graph if $n$ is an even number.

**Proof:** Since $n$ is even, units of $\mathbb{Z}_n$ must be odd. Thus, no two even labeled vertices are adjacent. This implies that the even labeled vertices and the odd labeled vertices form a bipartition of the vertex set. \qed

**Corollary 2.3.1.** If $n$ is even, then $X_n$ has no odd length cycles. In particular, $X_n$ is triangle-free.

### 3. Number of Triangles in $X_n$

Since $X_n$ is triangle-free for $n$ even, we will consider $n$ to be an odd number in this section. Let us denote by $\{a, b, c\}$ a triangle in $X_n$ with vertices $a$, $b$ and $c$; therefore $\pm(b-a)$, $\pm(c-b)$, $\pm(a-c)$ are elements in $U_n$. Without loss of generality we may assume that our triangles have vertices $\{0, 1, u\}$, $u \in U_n$. If we denote by $T_{01}$ the set of all the triangles having the common vertices 0 and 1 i.e., $T_{01} = \{\{0, 1, u\} | u \in U_n\}$, then the cardinality $|T_{01}| = |\{u \in U_n | (u-1) \in U_n\}|$. Using elementary counting procedures it can be proved that

$$|T_{01}| = n \prod_{p/n} \left(1 - \frac{2}{n}\right).$$

To count the number of triangles in $X_n$ let us consider the action of the group $G = U_n \times \mathbb{Z}_n$ on the set of triangles of $X_n$ i.e., if $(v, x) \in G$, then $(v, x)\{0, 1, u\} = \{vx, v(1+x), v(u+x)\}$. Each orbit of the triangles corresponding to the pair $(v, x)$ may have at most six different elements $(0, 0, 1), (0, u^{-1}, 1), (1-u^{-1}, 0, 1), (1, (u-1)u^{-1}, 0), (u(u-1)^{-1}, 1, 0)$ in $T_{01}$. It can happen that some orbits have exactly 3, 2 or even 1 elements. We have $|T_{01}| = \sum_{d/6} d h_d$, where $h_d$ is the number of orbits with exactly $d$ elements in $T_{01}$. The orbits with $d$ different elements have a number of triangles given by $|Oh_d| = \frac{n\phi(n)}{6d} = \frac{d(n)\phi(n)}{6}$. Thus, if $T$ stands for the number of triangles of $X_n$, then

$$T = \sum_{d/6} h_d|O_d| = h_6n\phi(n) + h_3\frac{n\phi(n)}{2} + h_2\frac{n\phi(n)}{3} + h_1\frac{n\phi(n)}{6},$$

$$T = n\phi(n)\left(h_6 + \frac{h_3}{2} + \frac{h_2}{3} + \frac{h_1}{6}\right) = n\phi(n)\frac{|T_{01}|}{6} = n\phi(n)\frac{\phi_2(n)}{6},$$

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where \( \phi_2(n) = n \sum_{p/n} \left(1 - \frac{2}{n}\right) \) by its resemblance to the Euler \( \phi \)-function. Note that \( \phi_2(n) \) is the number of consecutive units modulo \( n \).

Finally, if we denote by \( \IT \) the number of triangles in \( X_n \) that have two sides painted with equal colors, let us call these "isosceles-color-triangles" and denote by \( \ET \) the corresponding number of triangles with their three sides painted in different colors, called totally multicolored triangles. With obvious meaning of "scalene-color-triangles" we obtain the following result.

**Proposition 3.1.** If \( X_n \) is a Unitary Cayley graph with \( n \) an odd number then

\[
\IT = \frac{n\phi(n)}{2} \quad \text{and} \\
\ET = \frac{n\phi(n)}{6}(\phi_2(n) - 3).
\]

**Proof:** Since \( n \) is odd \( \{0, 1, -1\} \) is a isosceles color triangle; its orbit under the group action gives all the others. It can easily be verified that \( d = 3 \). Thus the formula for \( \IT \) follows.

For the scalene-color-triangles let us observe that there is no "equilateral-color-triangles", i.e. triangles with its three sides painted with the same color. Therefore, \( \ET = T - \IT \).

**Remark:** \( X_9 \) has 27 triangles and all of them are isosceles-color-triangles. The first graph having scalene-color-triangles for \( n \) different from a prime number is \( X_{21} \). It has 126 isosceles-color-triangles, 84 scalene-color-triangles and a total of 210 triangles.

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**References**

