On certain \( \mathcal{C} \)-ultrahomogeneous graphs obtained from cubic distance transitive graphs

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Abstract

The notion of a \( \mathcal{C} \)-ultrahomogeneous (or \( \mathcal{C} \)-UH) graph due to D. Isaksen et al. is adapted for digraphs and applied to the cubic distance-transitive graphs, considered both as graphs and digraphs, when \( \mathcal{C} \) is formed by shortest cycles and \((k-1)\)-paths, with \( k = \) arc-transitivity. Moreover, \((k-1)\)-powers of shortest cycles taken with orientation assignments that make these graphs become \( \mathcal{C} \)-UH digraphs are ‘zipped’ into novel \( \mathcal{C} \)-UH graphs, with \( \mathcal{C} \) formed by copies of \( K_3, K_4, C_7 \) and \( L(Q_3) \). In particular, the Biggs-Smith graph yields a connected edge-disjoint union of 102 copies of \( K_4 \) which is the non-line-graphical Menger graph of a self-dual \((102_4)\)-configuration, a \( K_7 \)-fastened \( \{K_4, L(Q_3)\} \)-UH graph. This stands in contrast with the self-dual \((42_4)\)-configuration of [7], whose non-line-graphical Menger graph is a \( K_2 \)-fastened \( \{K_4, K_{2,2,2}\} \)-UH graph.

Keywords: ultrahomogeneous graph; digraph; shortest cycle; arc-transitivity

1 Preliminaries

The study of ultrahomogeneous graphs (resp. digraphs) can be traced back to [15], [11] and [14]. Following a line of research initiated by [13], given a collection \( \mathcal{C} \) of (di)graphs closed under isomorphisms, a (di)graph \( G \) is said to be \( \mathcal{C} \)-ultrahomogeneous (or \( \mathcal{C} \)-UH) if every isomorphism between two induced members of \( \mathcal{C} \) in \( G \) extends to an automorphism of \( G \). If \( \mathcal{C} = \{H\} \) is the isomorphism class of a graph \( H \), we say that such a \( G \) is \( \{H\} \)-UH or \( H \)-UH. In [13], \( \mathcal{C} \)-UH graphs are defined and studied when \( \mathcal{C} \) is the collection of either (a) the complete graphs, or (b) the disjoint unions of complete graphs, or (c) the complements of those unions. In [7], a \( \{K_4, K_{2,2,2}\} \)-UH graph that fastens objects of (a) and (c), namely \( K_4 \) and \( K_{2,2,2} \), is presented.

We may consider a graph \( G \) as a digraph by considering each edge \( e \) of \( G \) as a pair of oppositely oriented (or OO) arcs \( e \) and \( (e)^{-1} \). Then, ‘zipping’ \( e \) and \( (e)^{-1} \) allows to recover \( e \), technique to be used repeatedly for graphs, below.
(In [8], however, a strongly connected $C_4$-UH oriented graph without OO arcs is presented).

Let $M$ be a sub(di)graph of a (di)graph $H$ and let $G$ be both an $M$-UH and an $H$-UH (di)graph. We say that $G$ is a (fastened) $(H;M)$-UH (di)graph if, given a copy $H_0$ of $H$ in $G$ containing a copy $M_0$ of $M$, then there exists exactly one copy $H_1 \neq H_0$ of $H$ in $G$ with $V(H_0) \cap V(H_1) = V(M_0)$ and $A(H_0) \cap A(H_1) = A(M_0)$, where $A(H_1)$ is formed by those arcs (of $e$) whose orientations are reversed with respect to those of the arcs $e$ of $A(H_1)$, and moreover: no more vertices or arcs than those in $M_0$ are shared by $H_0$ and $H_1$.

In the undirected case, the vertex and arc conditions above can be condensed as $H_0 \cap H_1 = M_0$, which is generalized by saying that a graph $G$ is an $\ell$-fastened $(H;M)$-UH graph if given a copy $H_0$ of $H$ in $G$ containing a copy $M_0$ of $M$, then there exist exactly $\ell$ copies $H_i \neq H_0$ of $H$ in $G$ such that $H_i \cap H_0 = M_0$, for each one of $i = 1, 2, \ldots, \ell$, and such that no more vertices or edges than those in $M_0$ are shared by each two of $H_0, H_1, \ldots, H_\ell$.

Let $H_i$ be a connected graph, for $i = 1, \ldots, h$. We say that a graph $G$ is a $K_2$-fastened $\{H_i\}_{i=1}^h$-UH graph if, for every $i = 1, \ldots, h$: (a) $G$ is an $H_i$-UH graph; (b) $G$ is representable as an edge-disjoint union of a number $n_i$ of induced copies of $H_i \subset H_j$, ($j \neq i$); (c) $G$ has a constant number $n_i$ of copies of $H_i$ incident at each vertex, with no two such copies sharing more than one vertex; (d) $G$ has exactly $n_i$ copies of $H_i$ as induced subgraphs isomorphic to $H_i$; (e) $G$ has each edge in exactly one copy of $H_i$.

Below, self-dual configurations and their Levi and Menger graphs are as in [6]. The objective of [7] was to present a self-dual $(42_4)$-configuration whose Menger graph $G$ is a non-line-graphical $K_2$-fastened $\{K_4, K_{2,2,2}\}$-UH graph; it has exactly 42 copies of $K_4$ and 21 copies of $K_{2,2,2}$, with four copies of $K_4$ and three copies of $K_{2,2,2}$ incident at each of its 42 vertices. This was relevant in view of the line graphs of the $d$-cubes, $3 \leq d \in \mathbb{Z}$, (for example the cuboctahedron $L(Q_3)$, which are $K_2$-fastened $\{K_d, K_{2,2,2}\}$-UH).

We will work with the cubic distance-transitive (or CDT) graphs $G$ (see [2, 4]):

<table>
<thead>
<tr>
<th>CDT graph $G$</th>
<th>$a$</th>
<th>$d$</th>
<th>$g$</th>
<th>$k$</th>
<th>$\eta$</th>
<th>$\alpha$</th>
<th>$h$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedral graph $K_4$</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
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<tr>
<td>Thomsen graph $K_{3,3}$</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>9</td>
<td>72</td>
<td>1</td>
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<tr>
<td>3-cube graph $Q_2$</td>
<td>8</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>6</td>
<td>45</td>
<td>0</td>
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</tr>
<tr>
<td>Petersen graph</td>
<td>10</td>
<td>2</td>
<td>5</td>
<td>3</td>
<td>12</td>
<td>120</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Heawood graph</td>
<td>14</td>
<td>3</td>
<td>6</td>
<td>4</td>
<td>28</td>
<td>336</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Pappus graph</td>
<td>18</td>
<td>4</td>
<td>6</td>
<td>3</td>
<td>18</td>
<td>216</td>
<td>1</td>
<td>3</td>
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<tr>
<td>Dodecahedral graph</td>
<td>20</td>
<td>5</td>
<td>5</td>
<td>2</td>
<td>12</td>
<td>120</td>
<td>0</td>
<td>1</td>
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<tr>
<td>Desargues graph</td>
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<td>5</td>
<td>6</td>
<td>3</td>
<td>20</td>
<td>240</td>
<td>1</td>
<td>1</td>
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<tr>
<td>Coxeter graph</td>
<td>28</td>
<td>4</td>
<td>7</td>
<td>3</td>
<td>24</td>
<td>336</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Tutte 8-cage</td>
<td>30</td>
<td>4</td>
<td>8</td>
<td>5</td>
<td>90</td>
<td>1440</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Foster graph</td>
<td>90</td>
<td>2</td>
<td>10</td>
<td>5</td>
<td>216</td>
<td>4320</td>
<td>1</td>
<td>0</td>
</tr>
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<td>Biggs-Smith graph</td>
<td>102</td>
<td>7</td>
<td>9</td>
<td>4</td>
<td>136</td>
<td>2448</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

where $a$, $d$, $g$, $k$, $\eta$ and $\alpha$ are order, diameter, girth, AT or arc-transitivity, number of $g$-cycles and number of automorphisms, respectively, with $b$ (resp. $h$) = 1 if $G$ is bipartite (resp. hamiltonian) and $= 0$ otherwise, and $\kappa$ defined as follows: let $P_k$ and $\bar{P}_k$ be respectively a $(k - 1)$-path and a directed $(k - 1)$-path (i.e., of length $k - 1 > 0$); let $C_g$ and $\bar{C}_g$ be respectively a cycle and a directed cycle
of length $g$; then (see Theorem 2 below): $\kappa = 0$, if $G$ is not $(\vec{C}_g; \vec{P}_k)$-UH; $\kappa = 1$, if $G$ is planar; $\kappa = 2$, if $G$ is $(\vec{C}_g; \vec{P}_k)$-UH with $g = 2(k - 1)$; $\kappa = 3$, if $G$ is $(\vec{C}_g; \vec{P}_k)$-UH with $g > 2(k - 1)$.

Given a finite graph $H$ and a subgraph $M$ of $H$ with $|V(H)| > 3$, we say that a graph $G$ is strongly fastened (or SF) $(H; M)$-UH if there is a descending sequence of connected subgraphs $M = M_1, \ldots, M_{|V(H)|-2} \equiv K_2$ such that: (a) $M_{i+1}$ is obtained from $M_i$ by the deletion of a vertex, for $i = 1, \ldots, |V(H)| - 3$ and (b) $G$ is a $(2^i - 1)$-fastened $(H; M_i)$-UH graph, for $i = 1, \ldots, |V(H)| - 2$.

Theorem 1 below asserts that every CDT graph is an SF $(C_g; P_k)$-UH graph.

Another SF $(H; M)$-UH graph appears in Theorem 9 (Section 3), for which is convenient to set the following definitions. A graph $G$ is $rK_n$-frequent if every edge $e$ of $G$ is the intersection in $G$ of exactly $r$ copies of $K_n$, and these have only $e$ and its endvertices in common. (For example, $K_4$ is $2K_3$-frequent; $L(Q_3)$ is $1K_3$-frequent). A graph $G$ is $K_3$-fastened $\{H_2, H_1\}$-UH, where $H_1$ is $iK_3$-frequent, $(i = 1, 2)$, if: (a) $G$ is an $H_2$-UH graph and an edge-disjoint union of copies of $H_2$; (b) $G$ is SF $(H_1; K_3)$-UH; (c) each copy of $H_2$ in $G$ has any of its subgraph copies of $K_3$ in common exactly with two copies of $H_1$ in $G$.

Given a graph $C$ and $0 < k \in \mathbb{Z}$ such that $k$ is at most the diameter of $C$, recall that the $k$-power graph $C^k$ of $C$ has $V(C^k) = V(C)$ and that two vertices are adjacent in $C^k$ if and only if they are at distance $k$ in $C$. Theorem 2 establishes which CDT graphs are $(\vec{C}_g; \vec{P}_k)$-UH digraphs. Elevating the resulting oriented cycles to the $(k - 1)$-power enables the construction, in Section 3, offastened $\mathcal{C}$-UH graphs, with $\mathcal{C}$ formed by copies of $K_3$, $K_4$, $C_7$ and $L(Q_3)$, when $\kappa = 3$, via ‘zipping’ of the OO induced $(k - 1)$-arcs shared (as $(k - 1)$-paths) by pairs of OO $g$-cycles. In particular: (a) the Pappus (resp. Desargues) graph yields the disjoint union of two copies of the Menger graph of the self-dual $(9_3)$- (resp. $(10_3)$-) configuration. [6]; (b) the Coxeter graph yields the Klein graph on 56 vertices, [3, 17]; (c) the Biggs-Smith graph yields the Menger graph of a self-dual $(102_4)$-configuration, a non-line-graphical $K_3$-fastened $\{K_4, L(Q_3)\}$-UH graph, in contrasts with the self-dual $(42_4)$-configuration of [7], whose Menger graph is $K_2$-fastened $\{K_4, K_{2,2,2}\}$-UH.

\section{$(C_g, P_k)$-UH properties of CDT graphs}

**Theorem 1** Let $G$ be a CDT graph of girth $g$ and $AT = k$. Then $G$ is an SF $(C_g; P_{k+i+2})$-UH graph, for $i = 0, 1, \ldots, k - 2$. In particular, $G$ is a $(C_g; P_k)$-UH graph and has exactly $2^{k-2}3ng^{-1}$ $g$-cycles.

\textbf{Proof.} We have to see that each CDT graph $G$ with girth $g$ and $AT = k$ is a $(2^i - 1)$-fastened $(C_g; P_{k-i})$-UH graph, for $i = 0, 1, \ldots, k - 2$. In fact, each $(k - i - 1)$-path $P = P_{k-i}$ of any such $G$ is shared exactly by $2^i$ $g$-cycles of $G$, for $i = 0, 1, \ldots, k - 2$. Moreover, each two of these $2^i$ $g$-cycles have just $P$ in common. This and a simple counting argument for the number of $g$-cycles, as cited in the table above, yield the assertions in the statement. \hfill $\square$
Theorem 2  The CDT graphs $G$ of girth $g$ and $AT = k$ that are not $(\vec{C}_g; \vec{P}_k)$-UH digraphs are the Petersen graph, the Heawood graph and the Foster graph. The remaining nine CDT graphs are fastened $(\vec{C}_g; \vec{P}_k)$-UH.

Proof. Given a $(\vec{C}_g; \vec{P}_k)$-UH graph $G$, an assignment of an orientation to each $g$-cycle of $G$ such that the two $g$-cycles shared by each $(k - 1)$-path receive opposite orientations yields a $(\vec{C}_g; \vec{P}_k)$-orientation assignment (or $(\vec{C}_g; \vec{P}_k)$-OA). The collection of $\eta$ oriented $g$-cycles corresponding to the $\eta$ $g$-cycles of $G$, for a particular $(\vec{C}_g; \vec{P}_k)$-OA will be called an $(\eta \vec{C}_g; \vec{P}_k)$-OAC.

The graph $G = K_4$ on vertex set $\{1, 2, 3, 0\}$ admits the $(4 \vec{C}_3; \vec{P}_2)$-OAC

$$\{(123), (210), (301), (032)\}.$$  

The graph $G = K_{3,3}$ obtained from $K_6$ (with vertex set $\{1, 2, 3, 4, 5, 0\}$) by deleting the edges of the triangles $(1, 3, 5)$ and $(2, 4, 0)$ admits the $(9 \vec{C}_4; \vec{P}_3)$-OAC

$$\{(1234), (3210), (4325), (1430), (2145), (0125), (5230), (0345), (5410)\}.$$  

The graph $G = Q_3$ with vertex set $\{0, \ldots, 7\}$ and edge set $\{01, 23, 45, 67, 02, 13, 46, 57, 04, 15, 26, 37\}$ admits the $(6 \vec{C}_4; \vec{P}_2)$-OAC

$$\{(0132), (1045), (3157), (2376), (0264), (4675)\}.$$  

If $G = Pet$ is the Petersen graph, then $G$ can be obtained from the disjoint union of the 5-cycles $\mu_{\infty} = (u_0u_1u_2u_3u_4)$ and $\nu_{\infty} = (v_0v_2v_4v_1v_3)$ by the addition of the edges $(u_x, v_x)$, for $x \in \mathbb{Z}_5$. Apart from the two 5-cycles given above, the other ten 5-cycles of $G$ can be denoted by $\mu_x = (u_{x-1}u_xu_{x+1}v_{x+1}v_x)$ and $\nu_x = (v_{x-2}v_xv_{x+1}u_{x+1}u_x)$, for each $x \in \mathbb{Z}_5$. Then the following sequence of alternating 6-cycles and 2-arcs starts and ends up with opposite orientations:

$$\mu_2^+(u_3u_2u_1)\mu_{\infty}^+(u_0u_1u_2)\mu_1^+(u_2v_2v_0)\nu_0^-(v_3u_3u_2)\mu_2^+,$$

where the upper indices $\pm$ indicate either a forward or backward selection of orientation and each 2-path is presented with the orientation of the previously cited 5-cycle but must be present in the next 5-cycle with its orientation reversed. Thus, $Pet$ cannot be a $(\vec{C}_5; \vec{P}_2)$-UH digraph.

For each positive integer $n$, let $I_n$ stand for the $n$-cycle $(0, 1, \ldots, n - 1)$, where $0, 1, \ldots, n - 1$ are considered as vertices. If $G = Hea$ is the Heawood graph, then $G$ can be obtained from $I_{14}$ by adding the edges $(2x, 5 + 2x)$, for $x \in \{1, \ldots, 7\}$ where operations are in $\mathbb{Z}_{14}$. The 28 6-cycles of $G$ include the following 7 6-cycles: $\gamma_x = (2x, 1 + 2x, 2 + 2x, 3 + 2x, 4 + 2x, 5 + 2x)$, where $x \in \mathbb{Z}_7$. Then the following sequence of alternating 6-cycles and 3-arcs starts and ends up with opposite orientations for $\gamma_0$:

$$\gamma_0^+(2345)\gamma_1^+(7654)\gamma_2^+(6789)\gamma_3^+(ba98)\gamma_4^+(abcd)\gamma_5^+ (10dc)\gamma_6^+(0123)\gamma_0^-,$$

(where tridecimal notation is used, up to $d = 13$). Thus, $Hea$ cannot be a $(\vec{C}_7; \vec{P}_4)$-UH digraph.
If $G = P_{ap}$ is the Pappus graph, then $G$ can be obtained from $I_{18}$ by adding the edges $(1 + 6x, 6 + 6x), (2 + 6x, 9 + 6x), (4 + 6x, 11 + 6x)$, for $x \in \{0, 1, 2\}$, where operations are mod 18. Then $G$ admits a $(18 \tilde{C}_0; \tilde{P}_5)$-OAC formed by the oriented 6-cycles $A_0 = (123456), B_0 = (321010), C_0 = (34bced), D_0 = (0165gh), E_0 = (4329ab)$, (where octodecimal notation is used, up to $h = 17$), the 6-cycles $A_x, B_x, C_x, D_x, E_x$ obtained by adding $6x$ mod 18 to (the integer representations of) the vertices of $A_0, B_0, C_0, D_0, E_0$, where $x \in \mathbb{Z}_5 \setminus \{0\}$, and finally the 6-cycles $F_0 = (23e89), F_1 = (hg54ba), F_2 = (61idc7)$.

If $G = D_{od}$ is the dodecahedral graph, then $G$ can be seen as a 2-covering graph of the Petersen graph $H$, where each vertex $u_x$, (resp., $v_x$), of $H$ is covered by two vertices $a_x, c_x$, (resp. $b_x, d_x$). This can be done so that a $(12 \tilde{C}_0; \tilde{P}_5)$OAC of $G$ is formed by the oriented 5-cycles $(a_0a_1a_2a_3a_4), (c_0c_2c_1c_0)$ and for each $x \in \mathbb{Z}_5$ also $(a_0d_0b_x-2d_x+a_{x+1})$ and $(d_xb_x+2c_x+2c_x-2b_{x-2})$.

If $G = D_{es}$ is the Desargues graph, then $G$ can be obtained from the 20-cycle $I_{20}$, with vertices $4x, 4x + 1, 4x + 2, 4x + 3$ redenoted alternatively $x_0, x_1, x_2, x_3$, respectively, for $x \in \{0, \ldots, 4\}$, by adding the edges $(x_3, (x + 2)_0)$ and $(x_1, (x + 2)_2)$, where operations are mod 5. Then $G$ admits a $(20 \tilde{C}_5; \tilde{P}_3)$-OAC formed by the oriented 6-cycles $A_x, B_x, C_x, D_x, E_x, F_x$, for $x \in \{0, \ldots, 4\}$, where

$$A_x = (x_0, x_1, x_2, (x + 1)_0, (x + 1)_3), \quad B_x = (x_0, x_1, (x + 1)_1, (x + 1)_2), \quad C_x = (x_0, x_1, (x + 1)_1, (x + 1)_3), \quad D_x = (x_0, x_1, (x + 1)_0, (x + 1)_3).$$

If $G = Cox$ is the Coxeter graph, then $G$ can be obtained from the three 7-cycles $(u_1u_2u_3u_4u_5u_6u_0), (v_1v_2v_3v_4v_5v_6v_0), (t_1t_2t_3t_4t_5t_6t_0)$ by adding a copy of $K_{1,3}$ with degree-3 vertex $x$ and degree-1 vertices $u_x, v_x, t_x$, for each $x \in \mathbb{Z}_7$. Then $Co_x$ admits the $(24 \tilde{C}_7; \tilde{P}_3)$-OAC

$$\begin{align*}
0^1 &= (u_1u_2u_3u_4u_5u_6u_0), & 0^2 &= (v_1v_2v_3v_4v_5v_6v_0), & 0^3 &= (t_1t_2t_3t_4t_5t_6t_0), \\
1^1 &= (u_1v_1u_2v_2u_3v_3u_4), & 1^2 &= (u_1v_1u_2v_2u_3v_3u_4), & 1^3 &= (u_1v_1u_2v_2u_3v_3u_4), \\
2^1 &= (u_1v_1u_2v_2u_3v_3u_4), & 2^2 &= (u_1v_1u_2v_2u_3v_3u_4), & 2^3 &= (u_1v_1u_2v_2u_3v_3u_4), \\
3^1 &= (u_1v_1u_2v_2u_3v_3u_4), & 3^2 &= (u_1v_1u_2v_2u_3v_3u_4), & 3^3 &= (u_1v_1u_2v_2u_3v_3u_4), \\
4^1 &= (u_1v_1u_2v_2u_3v_3u_4), & 4^2 &= (u_1v_1u_2v_2u_3v_3u_4), & 4^3 &= (u_1v_1u_2v_2u_3v_3u_4), \\
5^1 &= (u_1v_1u_2v_2u_3v_3u_4), & 5^2 &= (u_1v_1u_2v_2u_3v_3u_4), & 5^3 &= (u_1v_1u_2v_2u_3v_3u_4), \\
6^1 &= (u_1v_1u_2v_2u_3v_3u_4), & 6^2 &= (u_1v_1u_2v_2u_3v_3u_4), & 6^3 &= (u_1v_1u_2v_2u_3v_3u_4), \\
7^1 &= (u_1v_1u_2v_2u_3v_3u_4), & 7^2 &= (u_1v_1u_2v_2u_3v_3u_4), & 7^3 &= (u_1v_1u_2v_2u_3v_3u_4).
\end{align*}$$

If $G = Tut$ is Tutte’s 8-cage, then $G$ can be obtained form $I_{30}$, with vertices $5x, 5x + 1, 5x + 2, 5x + 3, 5x + 4, 5x + 5$ denoted alternatively $x_0, x_1, x_2, x_3, x_4, x_5$, respectively, for $x \in \mathbb{Z}_5$, by adding the edges $(x_5, (x + 2)_0), (x_1, (x + 1)_4)$ and $(x_2, (x + 2)_3)$. Then $G$ admits the $(90 \tilde{C}_8; \tilde{P}_5)$-OAC formed by the oriented 8-cycles

$$\begin{align*}
A^0 &= (4x_0, 0x_0, 0x_0, 0x_0), & B^0 &= (4x_0, 0x_0, 0x_0, 0x_0), & C^0 &= (0x_0, 0x_0, 0x_0, 0x_0), \\
D^0 &= (3x_0, 3x_0, 3x_0, 3x_0), & E^0 &= (4x_0, 4x_0, 4x_0, 4x_0), & F^0 &= (4x_0, 4x_0, 4x_0, 4x_0), \\
G^0 &= (1x_1, 2x_1, 2x_1, 2x_1), & H^0 &= (1x_1, 2x_1, 2x_1, 2x_1), & I^0 &= (1x_1, 2x_1, 2x_1, 2x_1), \\
P^0 &= (1x_0, 1x_0, 1x_0, 1x_0), & Q^0 &= (1x_0, 1x_0, 1x_0, 1x_0), & R^0 &= (1x_0, 1x_0, 1x_0, 1x_0),
\end{align*}$$

with together those obtained from these 18 8-cycles by adding $x \in \mathbb{Z}_5$ uniformly mod 5 to all subindices. Accordingly, these 8-cycles are denoted $A^x, \ldots, R^x$, where $x \in \mathbb{Z}_5$. 

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If \( G = \text{Fos} \) is the Foster graph, then \( G \) can be obtained from \( I_{90} \), with \( 5x, 5x + 1, 5x + 2, 5x + 3, 5x + 4, 5x + 5 \) denoted alternatively \( x_0, x_1, x_2, x_3, x_4, x_5 \), respectively, for \( x \in \mathbb{Z}_{15} \), by adding the edges \((x_4, (x + 2)_1)\), \((x_0, (x + 2)_{15})\) and \((x_2, (x + 6)_3)\). The 90 10-cycles of \( G \) include the following 15 10-cycles, where \( x \in \mathbb{Z}_{15} \).

\[
\phi_x = (x_4 x_6 (x+1)_1 (x+1)_2 (x+1)_3 (x+1)_4 (x+1)_5 (x+1)_6 (x+1)_7 (x+1)_8 (x+2)_9),
\]

Then the following sequence of alternating 10-cycles and 4-arcs:

\[
\phi^0_0[j] \phi^+_0[14] \phi^-_0[3] \phi^+_0[5] \phi^-_0[5] \phi^+_0[4] \phi^-_0[7] \phi^+_0[9] \phi^-_0[9] \phi^+_0[6] \phi^-_0[11] \phi^+_0[11] \phi^-_0[1] \phi^+_0[0] \phi^-_0[0]
\]

continues with \( \phi^+_0 \), of opposite orientation to that of the initial \( \phi^+_0 \), where \( [x_j] \) stands for a 3-path starting at the vertex \( x_j \) in the previously cited (to the left) oriented 10-cycle. Thus, \( \text{Fos} \) cannot be a \((\tilde{C}_9; \tilde{F}_2)\)-UH digraph.

If \( G = BS \) is the Biggs-Smith graph, then \( G \) can be reconstructed from four 17-cycles \( y = A, D, C, F \), namely \( A = (A_0, A_1, \ldots, A_9) \), \( D = (D_0, D_1, \ldots, D_9) \), \( C = (C_0, C_1, \ldots, C_9) \), \( F = (F_0, F_1, \ldots, F_9) \), (where each \( y \) has vertices \( y_i \) with \( i \) as an heptadecimal subindex, up to \( g = 16 \), advancing in 1,2,4,8 units mod 17 stepwise from left to right), by adding a 6-vertex tree with degree-1 vertices \( A_i, C_i, D_i, F_i \) and degree-2 vertices \( B_i, E_i \) and containing the 3-paths \( A_i B_i C_i \) and \( D_i E_i F_i \), for each \( i \in \mathbb{Z}_{17} \). Then \( G \) admits the \((102 \tilde{C}_9; \tilde{F}_2)\)-OAC formed by the oriented 9-cycles

\[
S^0 = (A_0 A_1 B_1 C_1 C_0 C_2 C_4 C_0 B_0), \quad T^0 = (C_0 C_1 C_2 C_4 A_1 A_2 A_1 A_3 B_0), \quad W^0 = (A_0 A_1 B_1 C_1 B_1 F_1 F_0 F_0 E_0 B_0), \quad X^0 = (C_0 C_4 B_4 E_4 D_4 D_2 D_0 E_0 B_0), \quad U^0 = (E_0 F_1 F_0 E_0 F_0 E_0 F_2 D_0 D_0), \quad Y^0 = (E_0 B_0 A_0 A_3 A_2 B_2 E_2 D_2 D_0), \quad V^0 = (E_0 D_0 D_2 D_0 D_2 D_0 D_0 E_0 F_0), \quad Z^0 = (F_0 F_0 E_0 E_0 C_6 C_0 A_0 B_0 E_0),
\]

together with those obtained from these eight 9-cycles by adding \( x \in \mathbb{Z}_{17} \) uniformly mod 17 to all subindices. Accordingly, these 9-cycles are denoted \( S^x, T^x, \) etc., where \( x \in \mathbb{Z}_{17} \).

3 ‘Zipping’ the \((k - 1)\)-powers of \( g \)-cycles ...

Given a CDT graph \( G \) with \( k = 3 \), consider the collection \( C_{g^k}^1(G) \) of \((k - 1)\)-powers of oriented \( g \)-cycles in the \((\eta \tilde{C}_g; \tilde{F}_k)\)-OAC of \( G \) in the proof of Theorem 2. If \( k = 3 \), then each arc \( \vec{e} \) of a member \( C^2 \) of \( C_{g^3}^1(G) \) is indicated by the middle vertex of the 2-arc \( \vec{E} \) in \( C \) for which \( \vec{e} \) stands, while the tail and head of \( \vec{e} \) are indicated by the tail and head of \( \vec{E} \), respectively. This is the case in Subsections 3.1-2 below, in which we consider the CDT graphs \( G \) with \( k = 3 \) in order to ‘zip’ such \( C^2 \)s along their OO arc pairs to obtain corresponding graphs \( Y(G) \) with \( C \)-UH properties. In Subsection 3.3, we consider a similar construction for \( k = 3 = k - 1 \), namely for the Biggs-Smith graph. In all these cases, the following sequence of operations is performed:

\[
G \rightarrow (\eta \tilde{C}_g; \tilde{F}_k)\text{-OAC}(G) \rightarrow C_{g^k}^1(G) \rightarrow Y(G).
\]
The CDT graphs $G$ with $\kappa = 0$ do not admit the approach suggested in the previous paragraph for their $g$-cycles lack a $(C_g; F_\kappa)$-OA; those with $\kappa = 1$ admit the approach with $Y(G) = G$ so nothing new is obtained more than a corresponding polyhedral graph (embeddable into the sphere) with faces delimited by $g$-cycles, namely the tetrahedral, 3-cube and dodecahedral graphs; those with $\kappa = 2$ again admit the approach, but since $\kappa/2 = k - 1$, then $Y(G) = (g - 1)G^{k-1}$, the multigraph of multiplicity $g - 1$ on the $(k - 1)$-th power of $G$.

### 3.1 for the Pappus, Desargues and $L(K_n)$ graphs, ...

If $G$ is either the Pappus graph $Pap$ or the Desargues graph $Des$, then $C_k^2(G)$ is formed by triangles conforming a graph $Y(G)$ with just two connected components $Y_1(G)$ and $Y_2(G)$.

![Diagram of toroidal cutouts of $Y_1(Pap)$ and $Y_2(Pap)$](image)

Each of $Y_1(Pap)$ and $Y_2(Pap)$ is embeddable in a closed orientable surface $T_1$ of genus 1, or 1-torus. In fact, Figure 1 shows toroidal cutouts of $Y_1(Pap)$ and $Y_2(Pap)$. Notice that the copies of $K_3$ in $C_k^2(G)$ are contractible in $T_1$. These triangles form two collections $\mathcal{H}_1, \mathcal{H}_2$ of copies $y_i^j$ of $K_3$ closed under parallel translation, where $y = A, B, C, D, E, F$; $i = 0, 1, 2$ and $j = 1, 2$, namely: the nine of $\mathcal{H}_1$ ($\mathcal{H}_2$) with horizontal edge below (above) its opposite vertex. There is also a collection $\mathcal{H}_0$ of nine non-contractible copies of $K_3$ in $G$, traceable linearly in three different parallel directions, three such triangles per direction, with the edges of each triangle indicated by an associated common vertex of $Pap$.

There are embeddings of $Y_1(Pap)$ and $Y_2(Pap)$ in $T_1$ for which $\mathcal{H}_0$ and either $\mathcal{H}_1$ or $\mathcal{H}_2$ provide the composing faces. In addition, each of $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_0$ is formed by three classes of parallel elements, in the sense that any two of them in such a class do not have vertices in common. The self-dual ($\theta_3$)-configuration in the following theorem is the Pappus configuration, [6].

**Theorem 3** $Y_1(Pap)$ and $Y_2(Pap)$ are isomorphic $K_2$-fastened $\{H_0, H_1, H_2\}$. 
UH graphs, where $H_i$ is a representative of $H_i$, for $i = 0, 1, 2$. Moreover, each of $Y_1(Pap)$ and $Y_2(Pap)$ can be taken as the Menger graph of the Pappus self-dual $(9_3)$-configuration in 12 different forms, by selecting the point set $P$ and the line set $L \neq \mathcal{P}$ so that $\{P, L\} \subset \{V(Pap), H_0, H_1, H_2\}$ and the incidence relation either as the inclusion of a vertex in a copy of $K_3$ or as the containment by a copy of $K_3$ of a vertex or as the sharing of an edge by two copies of $K_3$.

Proof. The statement can be established by managing the data given above. The 12 different claimed forms correspond to the arcs of the complete graph on vertex set $\{V(Pap), H_0, H_1, H_2\}$.

If $G = Des$, then $Y_1(G)$ and $Y_2(G)$ are isomorphic $K_2$-fastened $(K_4, K_3)$-UH graphs, each formed by five copies of $K_4$ and ten copies of $K_3$, with each such copy of $K_3$: (a) not forming part of a copy of $K_4$ in $Y_1(G)$ or $Y_2(G)$; (b) having its edges indicated with a constant symbol, as shown in Figure 2.

![Figure 2: Representations of $Y_1(Des)$ and $Y_2(Des)$](image)

Deleting a copy $H$ of $K_4$ from such $Y_i(Des)$ yields a copy of $K_{2,2,2}$, four of whose composing copies of $K_3$, with no common edges, are faces of corresponding copies of $K_4 \neq H$; the other four copies of $K_3$ are among the ten mentioned copies of $K_3$ in $G$. A realization of $Y_1(G)$ (or $Y_2(G)$) in 3-space can be obtained from a regular octahedron $O_3$ realizing the $K_{2,2,2}$ cited above via the midpoints of the four segments joining the barycenters of four edge-disjoint alternate triangles in $O_3$ to the barycenter of $O_3$ by constructing the tetrahedra determined by each of these alternate triangles and the nearest constructed midpoint, as well as the fifth central tetrahedron determined by the four midpoints.

By considering the barycenters of the resulting five tetrahedra and the segments joining them, a copy of $K_5$ in 3-space is obtained. The geometric line graph $L(K_5)$ it gives place to appears as a smaller version of $Y_1(G)$ (or $Y_2(G)$) contained in a octahedron $O_5 \subset O_3$. This procedure may be repeated indefinitely, generating a sequence of realizations of $Y_1(G)$ (or $Y_2(G)$) in 3-space. Since
posed by five copies of $K_1$ graphs are isomorphic to $L(K_5)$, whose complement is Pet, then this sequence yields a corresponding sequence of realizations of Pet in 3-space. We notice that the ten vertices and ten copies of $K_3$ of either $Y_i(Des)$ $(i = 1, 2)$ may be considered as the points and lines of the Desargues self-dual $(10_4)$ configuration, and that the Menger graph of this coincides with $Y_i(Des)$, [6]. Each vertex of $Y_i(Des)$ is the meeting vertex of two copies of $K_4$ and three copies of $K_3$ not forming part of a copy of $K_4$.

**Theorem 4** $Y_1(Des)$ and $Y_2(Des)$ are $K_2$-fastened $\{K_4, K_3\}$-UH graphs composed by five copies of $K_4$ and ten copies of $K_3$ each. Moreover, the ten vertices and ten copies of $K_3$ in either graph constitute the Desargues self-dual $(10_4)$ configuration, which has the graph itself as its Menger graph. Furthermore, both graphs are isomorphic to $L(K_5)$, whose complement is the Petersen graph.

Theorem 4 can be partly generalized by replacing $L(K_5)$ by $L(K_n)$ $(n \geq 4)$. This produces a $K_2$-fastened $\{K_{n-1}, K_3\}$-UH graph.

![Graph Image](image_url)

**Figure 3**: $\mathcal{F}$-colored $Cox$ and the three charts of the Klein graph $Y(Cox)$

**Theorem 5** The line graph $L(K_n)$, with $n \geq 4$, is a $K_2$-fastened $\{K_{n-1}, K_3\}$-UH graph with $n$ copies of $K_{n-1}$ and $\binom{n}{3}$ copies of $K_3$. 
Proof. We assume that each vertex of $K_n$ is taken as a color of edges of $L(K_n)$ under the following rule: Color all the edges between vertices of $L(K_n)$ representing edges incident to a vertex $v$ of $K_n$ with color $v$. Then, each triple of edge colors for $L(K_n)$ corresponds to the edges of a well determined copy of $K_3$ in $L(K_n)$. Thus, there are exactly $\binom{n}{3}$ copies of $K_3$ intervening in $L(K_n)$ looked upon as a $\{K_{n-1}, K_3\}$-UH graph. \hfill \Box

3.2 ... for the Coxeter graph ...

The Fano plane $\mathcal{F}$, with point set $J_7 = \{1, \ldots, 7\}$ and line set \{124, 235, 346, 457, 561, 672, 713\}, yields a coloring to the vertices and edges of $G = \text{Cox}$, represented on the upper left of Figure 3.

Observe that the colors of each vertex $v$ of $G$ and its three incident edges form a quadruple $q$ whose complement $\mathcal{F} \setminus q$ is a triangle of $\mathcal{F}$ used as a ‘customary’ vertex denomination for $v$, [12], page 69. Then: (a) the triple formed by the colors of the edges incident to each vertex of $G$ is a line of $\mathcal{F}$; (b) the color of each edge $e$ of $G$ together with the colors of the endvertices of $e$ form a line of $\mathcal{F}$. In this representation of $G$, the vertices $u_x$, $z_x$, $v_x$, $t_x$ in the proof of Theorem 2 are depicted concentrically from the outside to the inside, respectively, starting with $i = 1$, say, on the upper middle vertices with ‘customary’ vertex denominations 257, 134, 356, 567.

The squares $C^2$ of the $24$ 7-cycles of $G$ are taken with the following (cyclically presented) orientations, where each vertex $v$ (resp. edge $e$) of a $C^2$ is indicated by the color of $v$ (resp., subindicated by the color of the middle vertex of the 2-path of $C$ that $e$ represents). In fact, the resulting oriented 7-cycles can be denoted $i^j$, where $i \in \{0\} \cup J_7$ and $j \in J_3 = \{1, 2, 3\}$, namely:

- $0^1$: $(1\,2\,3\,4\,5\,6\,7)$
- $1^1$: $(1\,2\,3\,4\,5\,6\,7)$
- $2^1$: $(1\,2\,3\,4\,5\,6\,7)$
- $3^1$: $(1\,2\,3\,4\,5\,6\,7)$
- $4^1$: $(1\,2\,3\,4\,5\,6\,7)$
- $5^1$: $(1\,2\,3\,4\,5\,6\,7)$
- $6^1$: $(1\,2\,3\,4\,5\,6\,7)$
- $7^1$: $(1\,2\,3\,4\,5\,6\,7)$

Each 2-arc of $G$ is cited exactly once in the oriented cycles $i^j$. Each 2-path of $G$ appears twice in the $i^j$s, once for each one of its two 2-arcs. The assumed orientation of each $C^2$ corresponds with the orientation of the corresponding 7-cycle $C$. Thus, we consider each 7-cycle $C$ with the orientation it induces in the corresponding $C^2$, say $i^j$. In this case, we denote such a $C$ as $i^j$.

Each 2-path $e$ of $G$ separates two of the 24 7-cycles of $G$, say $i^j$ and $k^j$, with their orientations opposite over $\mathcal{C}$. Now, $i^j$ and $k^j$ restrict to the two different 2-arcs provided by $\mathcal{C}$, say 2-arcs $e_1$ and $e_2$. Then, $e_1$ and $e_2$ represent corresponding arcs $\mathcal{C}_1$ and $\mathcal{C}_2$ in $i^j$ and $k^j$, respectively.

Let us see that $\mathcal{C}_1$ and $\mathcal{C}_2$ can be ‘zipped’ into an edge $e$ in the graph $Y(G)$. This, which happens to be the Klein graph, [3], graph 56B, can be assembled from the three charts shown on the upper right and bottom of Figure 3 by
'zipping' the 7-cycles \(i^j\), interpreted all with counterclockwise orientation. Each of these three charts conforms a 'rosette', where the 7-cycles \(i^j\) with \(i \neq 0\) are represented as 'petals' of the 'central' 7-cycles \(0^1, 0^2\) and \(0^3\). (The assemblage of \(Y(G)\) can be done also around the 7-cycles \(i^1, i^2\) and \(i^3\), for any \(i \in J_7\), not just \(i = 0\)). Moreover, each edge \(e\) in the external border of one of the three charts is accompanied by the denomination of a 7-cycle \(i^j\) incident to \(e\) which is a petal in one of the other two rosettes. Thus, the three charts can be assembled into the claimed graph \(Y(G)\). Moreover, the 24 7-cycles \(i^j\) can be filled each with a corresponding 2-cell, so because of the cancelations of the two opposite arcs on each edge of \(Y(G)\) (for having opposite orientations makes them mutually cancelable), \(Y(G)\) becomes embedded into a closed orientable surface \(T_3\). As for the genus of \(T_3\), observe that

\[
|V(Y(G))| = 2 \times 28 = 56 \quad \text{and} \quad |E(Y(G))| = 2|E(G)| = 2 \times 42 = 84,
\]

so that by the Euler characteristic formula for \(T_3\) here,

\[
|V(Y(G))| - |E(Y(G))| + |F_7(Y(G))| = 56 - 84 + 24 = -4 = 2 - 2g(T),
\]

and thus \(g = 3\), so \(T_3\) is a 3-torus. This yields the Klein map (of Coxeter notation) \(\{7,3\}_8\), (see [17]; note: the Petrie polygons of this map are 8-cycles).

**Theorem 6** The Klein graph \(Y(Cox)\) is a \((C_7; P_2)\)-UH graph composed by 24 7-cycles that yield the Klein map \(\{7,3\}_8\) in \(T_3\). \(\square\)

For the Klein map \(\{7,3\}_8\), the 3-torus appeared originally dressed as the Klein quartic \(x^3y + y^3z + z^3x = 0\), a Riemann surface and the most symmetrical curve of genus 3 over the complex numbers. The automorphism group for this Klein map is \(PSL(2,7) = GL(3,2)\), the same group as for \(F\), whose index is 2 in the automorphism groups of \(Hea, Cox\) and \(Y(Cox)\).

**Corollary 7** The graph \(Y'(Cox)\) whose vertices are the 7-cycles \(i^j\) of \(Y(Cox)\), with adjacency between two vertices if their representative 7-cycles have an edge in common, is regular of degree 7, chromatic number 8 and has a natural triangular \(T_3\)-embedding yielding the dual Klein map \(\{3,7\}_8\).

**Proof.** Each vertex \(i^j\) of \(Y'(Cox)\) is assigned color \(i \in \{0\} \cup J_7\). Also, we have a partition of \(T_3\) into 24 connected regions, each region having exactly seven neighboring regions, with eight colors needed for a proper map coloring. \(\square\)

### 3.3 … and for the Biggs-Smith graph

The cubes \(C^3\) of the 136 9-cycles \(C\) of the Biggs-Smith graph \(G = BS\) are formed by three disjoint 3-cycles each, yielding a total of \(3 \times 136 = 408\) 3-cycles. In fact, the \((136 \tilde{C}_3; \tilde{P}_2)\)-OAC mentioned in the proof of Theorem 2 for \(G\) determines a \((408 \tilde{C}_3; \tilde{P}_2)\)-OAC for \(Y(G)\). The resulting oriented 3-cycles are 'zipped' along
the pairs of OO copies of \( \vec{P}_2 \) obtained as cubes of OO copies of \( \vec{P}_4 \) in \( G \). The resulting ‘zipping’ of OO arcs yields 102 copies of \( K_4 \). These can be subdivided into six subcollections \{y\} of 17 copies each, where \( y \in \{A, B, C, D, E, F\} \) and \( i \in \{0, 1, \ldots, 16 = g\} = \mathbb{Z}_{17} \). The vertex sets \( V(y^i) \) of these copies of \( K_4 \), followed each by the corresponding set \( \Lambda(y_i) \) of copies of \( K_4 \) containing the vertex \( y_i \), are as follows:

\[
\begin{align*}
V(A^x) &= \{C_x, D_x, E_{x+4}, E_{x-4}\}, \quad \Lambda(A_4) = \{C^x, D^x, E^x, E^{x+4}\}; \\
V(B^x) &= \{D_{x-8}, D_{x-1}, F_x, F_{x+7}\}, \quad \Lambda(B_8) = \{D^{x-2}, D^{x-1}, F^x, F^{x+1}\}; \\
V(C^x) &= \{A_x, F_x, E_{x+2}, E_{x-1}\}, \quad \Lambda(C_8) = \{A^x, F^{x+8}, E^{x+4}, E^{x-1}\}; \\
V(D^x) &= \{A_x, D_x, B_{x+2}, B_{x-2}\}, \quad \Lambda(D_8) = \{A^x, D^x, B_{x+2}, B_{x-2}\}; \\
V(E^x) &= \{A_x, A_{x+2}, C_{x+4}, C_{x-4}\}, \quad \Lambda(E_8) = \{A^x, A^{x+4}, C^x, C^{x-4}\}; \\
V(F^x) &= \{C_{x-8}, F_{x-8}, B_x, B_{x+1}\}, \quad \Lambda(F_8) = \{C^x, F^{x+8}, B^x, B_{x+7}\};
\end{align*}
\]

where \( x \in \mathbb{Z}_{17} \). This presentation emphasizes a duality property existing between the vertices of \( G \) and the copies of \( K_4 \) in the cubes of the 9-cycles of \( G \).

A dual map realizing this duality property is given by an isomorphism from \( G \) (as presented in the proof of Theorem 2) onto a graph \( G' \) isomorphic to \( G \) and that can be obtained similarly from the four 17-cycles \( A' = (A^0, A^3, \ldots, A^x), D' = (D^0, D^3, \ldots, D^x), C' = (C^0, C^3, \ldots, C^x), F' = (F^0, F^3, \ldots, F^x) \) (which advance the vertex subindices in \( 3 = 1 \times 3, 7 = 2 \times (-5), 12 = 4 \times 3, 11 = 8 \times (-5) \) units mod 17 stepwise from left to right, respectively) by adding a 6-vertex tree with degree-1 vertices \( A^{3x}, C^{3x}, D^{-5x}, F^{8-5x} \) and degree-2 vertices \( B^5, E^{10-6x} \) and containing the 3-paths \( A^3B^{5-7i}C^{3i} \) and \( D^{-5x}E^{10-6x}F^{8-5x} \), for each \( x \in \mathbb{Z}_{17} \).

The cubes of the oriented cycles in the proof of Theorem 2 are:

\[
\begin{align*}
S_0^+ &\rightarrow (E_0^0 \setminus A_0) = (A_0, C_0, B_0), \quad E_0^0 \setminus A_0 = (A_1, C_5, B_0), \quad F_0^0 \setminus F_0 = (B_1, C_0, B_0); \\
T_0^+ &\rightarrow (E_4^0 \setminus C_0 = (C_0, A_4, A_1), \quad E_3^1 = (C_4, A_3, A_0), \quad D_0^0 \setminus D_2 = (B_4, A_2, B_0); \\
U_0^+ &\rightarrow (C_4^0 \setminus A_1 = (B_0, F_1, E_2), \quad B^x = (F_0, F_3, D_2), \quad B^0 = (F_0, F_3, D_2)); \\
V_0^+ &\rightarrow (A^0 \setminus C_2 = (E_0, D_1, E_2), \quad B^0 = (D_0, D_1, E_2), \quad B^0 = (D_0, D_1, E_2)); \\
W_0^+ &\rightarrow (C^0 \setminus E_0 = (E_0, C_5, E_0), \quad C^0 \setminus E_0 = (E_1, C_5, E_0), \quad F^0 = (C_0 = (B_1, C_0, B_0)); \\
X_0^+ &\rightarrow (A^0 \setminus E_2 = (C_0, E_4, D_0), \quad A^1 \setminus E_6 = (C_4, D_4, E_0), \quad D_2 = (B_4, D_2, B_0); \\
Y_0^+ &\rightarrow (C^1 \setminus F_1 = (E_0, A_1, E_2), \quad D^2 = (B_0, A_2, D_2), \quad D^2 = (B_0, A_2, D_2)); \\
Z_0^+ &\rightarrow (F^0 \setminus B_0 = (F_0, B_3, C_0), \quad F^0 = (F_0, C_3, B_0), \quad A^3 = (E_3, C_4, B_0)).
\end{align*}
\]

Because of the properties of \( G \), it can be seen that \( Y(G) \) is a \( K_4 \)-UH graph. Moreover, the vertices and copies of \( K_4 \) in \( Y(G) \) are the points and lines of a self-dual (102)-configuration which in turn has \( Y(G) \) as its Menger graph. (Compare with [6, 7]). However, in view of Beineke’s characterization of line graphs [1], and observing that \( Y(G) \) contains induced copies of \( K_{1,3} \), which are forbidden for line graphs of simple graphs, we conclude that \( Y(G) \) is non-line-graphical, as commented above for the graph treated in [7], the Menger graph of a self-dual (42)-configuration.

**Theorem 8** \( Y(BS) \) is an edge-disjoint union of 102 copies of \( K_4 \) with four such copies incident to each vertex. Moreover, \( Y(BS) \) is a non-line-graphical \( K_4 \)-UH graph. Its vertices and copies of \( K_4 \) are the points and lines, respectively, of a self-dual (102)-configuration, which in turn has \( Y(BS) \) as its Menger graph. This is an arc-transitive graph with regular degree 12, diameter 3, distance distribution (1, 12, 78, 11) and automorphism-group order 2448. Its associated Levi
The graph is a 2-arc-transitive graph with regular degree 4, diameter 6, distance distribution (1, 4, 12, 36, 78, 62, 11) and automorphism-group order 4896.

![Figure 4: Copy $\overrightarrow{Y}$ of $L(Q_3)$ in $Y(BS)$](image)

Any of the 102 copies of $K_4$ in $Y$ arises from the cubes of four of the 102 9-cycles of $G$. The subgraph of $G$ spanned by these four 9-cycles contains four degree-three vertices, (exactly those described in the data given above), and twelve degree-two vertices, (exactly the two middle vertices in each path of length 3 realizing an edge of such a copy of $K_4$). These twelve vertices form a copy $\mathcal{L}$ of $L(Q_3)$ in $Y(G)$. For the copy $A^0$ of $K_4$ in $Y(G)$, the copy $\overrightarrow{Y}$ of $L(Q_3)$ in $Y(G)$ accompanying such $\mathcal{L}$ can be represented as in Figure 4, where:

(a) the leftmost and rightmost dotted lines are identified by parallel translation;

(b) each of the eight copies of $K_3$ forms part of a corresponding copy of $K_4$ (among the 102 in $Y$) cited externally by name about its horizontal edge, with the fourth vertex cited internally. By presenting the elements of the shown representation orderly, we may indicate the copies $y^0$ of $L(Q_3)$ as follows, for $y = A, B, C, D, E, F$:

\[ \overrightarrow{\mathcal{Q}}: (B_0B_1F_4F_dD_1) (F_4D_4E_0C_4) D_2C_2D_4B_4 (D^2A_2F^4C_2B_8E_6) (B^4F_8A^4E_4E_8F_8B_8C_8) \]

\[ \overrightarrow{\mathcal{P}}: (D_0D_4E_7D_8) (D_2D_7D_6F_6) (D_1E_2D_4E_5) (B^4D_6C_8A^4A_4C_8B_2F_2) (B^4F_8A^4C_8E_4A_4B_0D_1) \]

\[ \overrightarrow{\mathcal{O}}: (A_1B_4F_8F_1) (F_2F_8B_1A_1) (D_1B_8D_6E_6) (D^2F_8C_8B^4D_8C_4E_2) (B^2D_7F^2C_8D^2B_4C_8E_7) \]

\[ \overrightarrow{\mathcal{N}}: (A_1E_2D_1A_1) (E_1A_3A_2D_2) (E_4C_4D_6D_7B_4E_8C_1) (C^4F_6A^4F_4D^4D_4E_4E_6) \]

\[ \overrightarrow{\mathcal{M}}: (A_1C_5B_2A_1) (A_4A_4B_1C_1) C_4B_8B_6C_6 (E^4A^4F^4F_4D^4D_4E_4E_2) (B^4C_8D^4D_8E^4B_4E_4E_1) \]

\[ \overrightarrow{\mathcal{L}}: (A_0C_1B_4F_8) (E_2E_6A_1C_1) C_4C_8E_2E_6 (E^4A^4F^4B^4B_4C^4E_4) (A^4D_8E^4C_4E_2B_4E_4E_1) \]

and obtain the remaining $y^0$ by uniform translations mod 17, for any $i \in \mathbb{Z}_{17}$.

Each vertex of $Y(G)$ belongs exactly to twelve such $\mathcal{L}$s. Figure 5 shows the complements of $A_0$ in four of the twelve copies of $L(Q_3)$ containing it (sharing the long vertical edges), where the black vertices form the 4-cycles containing $A_0$, and some edges and vertices appear repeated twice, once per copy of $L(Q_3)$; for example, the leftmost and rightmost edges must be identified by parallel translation; alternate internal anti-diagonal 2-paths in the figure also coincide with their directions reversed; (notice that the middle vertices of these 2-paths are the neighbors of $A_0$ in $G$, and that their degree-1 vertices are at distance 2 from $A_0$, again in $G$). The oriented 9-cycles of the $(\eta\mathcal{C}_5; \mathcal{P}_4)$-OAC of $G$ cited in the proof of Theorem 2 intervene, as indicated on the figure, in the formation of the oriented 3-cycles and copies of $L(Q_3)$ induced by the long vertical edges $(C_4C_1, A_3C_4, D_0B_2, E_1F_0)$ resp. for $E^0$, $E^3$, $D^0$, $C^0$) and the 6-cycles they separate $((D_0B_2A_3C_4A_2C_1), (A_3E_4D_0B_2D_4C_4), (D_0F_2E_1F_0F_6B_2), (E_1E_8C_4C_1B_0F_0)$ resp. for $E^4$, $D^2$, $C^1$, $F^0$).
Given a copy of $K_4$ and $L(Q_3)$ that contain $A_0$:

The information in the figure can be set as in the following arrangement, followed by two additional arrangements that complete all the information provided by the copies of $K_4$ and $L(Q_3)$ that contain $A_0$:

(Some edges are shared by two different of these three arrangements. In fact, each of the edges bordering the 2-paths mentioned above in anti-diagonal 4-paths is present also in the second or third arrangement. For example, the edge $B_1A_3$ of $E^4$ on the figure appears in the second arrangement).

In the same way, the vertices $B_0$, $C_0$, $D_0$, $E_0$ and $F_0$ admit similar arrangements (see the Appendix) and additions mod 17 yield the remaining information for neighboring copies of $K_4$ and $L(Q_3)$ at each vertex of $Y(G)$.

**Theorem 9** The graph $Y(BS)$ is an SF $(L(Q_3); K_3)$-UH graph. Moreover, each two copies of $L(Q_3)$ sharing a copy $H$ of $K_3$ in $Y(BS)$ also share $H$ with exactly one copy of $K_4$ in $Y(BS)$. Furthermore, each 4-cycle of $Y(BS)$ exists in just one copy of $L(Q_3)$ in $G$. Thus, $Y(BS)$ is $K_3$-fastened $\{K_4, L(Q_3)\}$-UH.

**Proof.** Given a copy $H$ of $L(Q_3)$ in $Y(BS)$ and a copy $\Delta$ of $K_3$ in $H$, there exists a unique copy $\neq H$ of $L(Q_3)$ that shares with $H$ the subgraph $\Delta$. In addition, any edge of $H$ is shared by exactly three other copies $H'$ $\neq H$, $H''$ $\neq H$ and $H'''$ $\neq H$ of $L(Q_3)$. Because of the symmetry reigning between the copies of $K_4$ and of $L(Q_3)$ in $Y(BS)$, the statement follows. □
3.4 Appendix: local behavior of $Y(BS)$

Here is the data that must replace the symbols in Figure 5 in order to yield the complements of $y_i$ in the copies of $K_4$ and $L(Q_3)$ incident to $y_0$, for $y = A, B, C, D, E, F$, where the rows are cited in parentheses that are preceded by $y_0$ and the fourth rows cite each appearing vertex just once:

\[
\begin{align*}
& (C_i B_i A_i C_i D_i E_i F_i G_i H_i) \\
& (C_i B_i A_i C_i D_i E_i F_i G_i H_i) \\
& (C_i B_i A_i C_i D_i E_i F_i G_i H_i) \\
& (C_i B_i A_i C_i D_i E_i F_i G_i H_i)
\end{align*}
\]

References


