

# On certain $\mathcal{C}$ -ultrahomogeneous graphs obtained from cubic distance transitive graphs

Italo J. Dejter  
University of Puerto Rico  
Rio Piedras, PR 00931-3355  
idejter@uprrp.edu

## Abstract

The notion of a  $\mathcal{C}$ -ultrahomogeneous (or  $\mathcal{C}$ -UH) graph due to D. Isaksen et al. is adapted for digraphs and applied to the cubic distance-transitive graphs, considered both as graphs and digraphs, when  $\mathcal{C}$  is formed by shortest cycles and  $(k - 1)$ -paths, with  $k =$  arc-transitivity. Moreover,  $(k - 1)$ -powers of shortest cycles taken with orientation assignments that make these graphs become  $\mathcal{C}$ -UH digraphs are ‘zipped’ into novel  $\mathcal{C}$ -UH graphs, with  $\mathcal{C}$  formed by copies of  $K_3$ ,  $K_4$ ,  $C_7$  and  $L(Q_3)$ . In particular, the Biggs-Smith graph yields a connected edge-disjoint union of 102 copies of  $K_4$  which is the non-line-graphical Menger graph of a self-dual  $(102_4)$ -configuration, a  $K_3$ -fastened  $\{K_4, L(Q_3)\}$ -UH graph. This stands in contrast with the self-dual  $(42_4)$ -configuration of [7], whose non-line-graphical Menger graph is a  $K_2$ -fastened  $\{K_4, K_{2,2,2}\}$ -UH graph.

**Keywords:** ultrahomogeneous graph; digraph; shortest cycle; arc-transitivity

## 1 Preliminaries

The study of ultrahomogeneous graphs (resp. digraphs) can be traced back to [16], [11] and [15], (resp. [9], [14] and [5]). Following a line of research initiated by [13], given a collection  $\mathcal{C}$  of (di)graphs closed under isomorphisms, a (di)graph  $G$  is said to be  $\mathcal{C}$ -ultrahomogeneous (or  $\mathcal{C}$ -UH) if every isomorphism between two induced members of  $\mathcal{C}$  in  $G$  extends to an automorphism of  $G$ . If  $\mathcal{C} = \{H\}$  is the isomorphism class of a graph  $H$ , we say that such a  $G$  is  $\{H\}$ -UH or  $H$ -UH. In [13],  $\mathcal{C}$ -UH graphs are defined and studied when  $\mathcal{C}$  is the collection of either **(a)** the complete graphs, or **(b)** the disjoint unions of complete graphs, or **(c)** the complements of those unions. In [7], a  $\{K_4, K_{2,2,2}\}$ -UH graph that fastens objects of (a) and (c), namely  $K_4$  and  $K_{2,2,2}$ , is presented.

We may consider a graph  $G$  as a digraph by considering each edge  $e$  of  $G$  as a pair of oppositely oriented (or OO) arcs  $\vec{e}$  and  $(\vec{e})^{-1}$ . Then, ‘zipping’  $\vec{e}$  and  $(\vec{e})^{-1}$  allows to recover  $e$ , technique to be used repeatedly for graphs, below.

(In [8], however, a strongly connected  $C_4$ -UH oriented graph without OO arcs is presented).

Let  $M$  be a sub(di)graph of a (di)graph  $H$  and let  $G$  be both an  $M$ -UH and an  $H$ -UH (di)graph. We say that  $G$  is a (fastened)  $(H; M)$ -UH (di)graph if given a copy  $H_0$  of  $H$  in  $G$  containing a copy  $M_0$  of  $M$ , there exists exactly one copy  $H_1 \neq H_0$  of  $H$  in  $G$  with  $V(H_0) \cap V(H_1) = V(M_0)$  and  $A(H_0) \cap \bar{A}(H_1) = A(M_0)$ , where  $\bar{A}(H_1)$  is formed by those arcs  $(\bar{e})^{-1}$  whose orientations are reversed with respect to those of the arcs  $\bar{e}$  of  $A(H_1)$ , and moreover: no more vertices or arcs than those in  $M_0$  are shared by  $H_0$  and  $H_1$ . In the undirected case, the vertex and arc conditions above are condensed as  $H_0 \cap H_1 = M_0$ , which is generalized by saying that a graph  $G$  is an  $\ell$ -fastened  $(H; M)$ -UH graph if given a copy  $H_0$  of  $H$  in  $G$  containing a copy  $M_0$  of  $M$ , then there exist exactly  $\ell$  copies  $H_i \neq H_0$  of  $H$  in  $G$  such that  $H_i \cap H_0 = M_0$ , for  $i = 1, 2, \dots, \ell$ , and such that no more vertices or edges than those in  $M_0$  are shared by each two of  $H_0, H_1, \dots, H_\ell$ .

Let  $H_i$  be a connected graph, for  $i = 1, \dots, h$ . We say that a graph  $G$  is a  $K_2$ -fastened  $\{H_i\}_{i=1}^h$ -UH graph if, for every  $i = 1, \dots, h$ : **(a)**  $G$  is an  $H_i$ -UH graph; **(b)**  $G$  is representable as an edge-disjoint union of a number  $n_i$  of induced copies of  $H_i \not\subset H_j$ , ( $j \neq i$ ); **(c)**  $G$  has a constant number  $m_i$  of copies of  $H_i$  incident at each vertex, with no two such copies sharing more than one vertex; **(d)**  $G$  has exactly  $n_i$  copies of  $H_i$  as induced subgraphs isomorphic to  $H_i$ ; **(e)**  $G$  has each edge in exactly one copy of  $H_i$ .

Below, self-dual configurations and their Levi and Menger graphs are as treated in [6]. The objective of [7] was to present a self-dual  $(42_4)$ -configuration whose Menger graph  $G$  is a non-line-graphical  $K_2$ -fastened  $\{K_4, K_{2,2,2}\}$ -UH graph; it has exactly 42 copies of  $K_4$  and 21 copies of  $K_{2,2,2}$ , with four copies of  $K_4$  and three copies of  $K_{2,2,2}$  incident at each of its 42 vertices. This was relevant in view of the line graphs of the  $d$ -cubes,  $3 \leq d \in \mathbf{Z}$ , (for example the cuboctahedron  $L(Q_3)$ ), which are  $K_2$ -fastened  $\{K_d, K_{2,2}\}$ -UH.

We will work with the cubic distance-transitive (or CDT) graphs  $G$  (see [2, 4]):

CDT graph $G$	$n$	$d$	$g$	$k$	$\eta$	$a$	$b$	$h$	$\kappa$
Tetrahedral graph $K_4$	4	1	3	2	4	24	0	1	1
Thomsen graph $K_{3,3}$	6	2	4	3	9	72	1	1	2
3-cube graph $Q_3$	8	3	4	2	6	48	1	1	1
Petersen graph	10	2	5	3	12	120	0	0	0
Heawood graph	14	3	6	4	28	336	1	1	0
Pappus graph	18	4	6	3	18	216	1	1	3
Dodecahedral graph	20	5	5	2	12	120	0	1	1
Desargues graph	20	5	6	3	20	240	1	1	3
Coxeter graph	28	4	7	3	24	336	0	0	3
Tutte 8-cage	30	4	8	5	90	1440	1	1	2
Foster graph	90	8	10	5	216	4320	1	1	0
Biggs-Smith graph	102	7	9	4	136	2448	0	1	3

where  $n, d, g, k, \eta$  and  $a$  are order, diameter, girth, AT or arc-transitivity, number of  $g$ -cycles and number of automorphisms, respectively, with  $b$  (resp.  $h$ ) = 1 if  $G$  is bipartite (resp. hamiltonian) and = 0 otherwise, and  $\kappa$  defined as follows: let  $P_k$  and  $\vec{P}_k$  be respectively a  $(k-1)$ -path and a directed  $(k-1)$ -path (i.e., of length  $k-1 > 0$ ); let  $C_g$  and  $\vec{C}_g$  be respectively a cycle and a directed cycle of length  $g$ ; then (see Theorem 2 below):  $\kappa = 0$ , if  $G$  is not  $(\vec{C}_g; \vec{P}_k)$ -UH;  $\kappa = 1$ ,

if  $G$  is planar;  $\kappa = 2$ , if  $G$  is  $(\vec{C}_g; \vec{P}_k)$ -UH with  $g = 2(k - 1)$ ;  $\kappa = 3$ , if  $G$  is  $(\vec{C}_g; \vec{P}_k)$ -UH with  $g > 2(k - 1)$ .

Given a finite graph  $H$  and a subgraph  $M$  of  $H$  with  $|V(H)| > 3$ , we say that a graph  $G$  is *strongly fastened* (or SF)  $(H; M)$ -UH if there is a descending sequence of connected subgraphs  $M = M_1, \dots, M_{|V(H)|-2} \equiv K_2$  such that: **(a)**  $M_{i+1}$  is obtained from  $M_i$  by the deletion of a vertex, for  $i = 1, \dots, |V(H)| - 3$  and **(b)**  $G$  is a  $(2^i - 1)$ -fastened  $(H; M_i)$ -UH graph, for  $i = 1, \dots, |V(H)| - 2$ . Theorem 1 below asserts that every CDT graph is an SF  $(C_g; P_k)$ -UH graph.

Another SF  $(H; M)$ -UH graph appears in Theorem 9 (Section 3), for which is convenient to set the following definitions. A graph  $G$  is  $rK_s$ -frequent if every edge  $e$  of  $G$  is the intersection in  $G$  of exactly  $r$  copies of  $K_s$ , and these have only  $e$  and its endvertices in common. (For example,  $K_4$  is  $2K_3$ -frequent;  $L(Q_3)$  is  $1K_3$ -frequent). A graph  $G$  is  $K_3$ -fastened  $\{H_2, H_1\}$ -UH, where  $H_i$  is  $iK_3$ -frequent, ( $i = 1, 2$ ), if: **(a)**  $G$  is an  $H_2$ -UH graph and an edge-disjoint union of copies of  $H_2$ ; **(b)**  $G$  is SF  $(H_1; K_3)$ -UH; **(c)** each copy of  $H_2$  in  $G$  has any of its subgraph copies of  $K_3$  in common exactly with two copies of  $H_1$  in  $G$ .

Given a graph  $C$  and  $0 < k \in \mathbf{Z}$  such that  $k$  is at most the diameter of  $C$ , recall that the  $k$ -power graph  $C^k$  of  $C$  has  $V(C^k) = V(C)$  and that two vertices are adjacent in  $C^k$  if and only if they are at distance  $k$  in  $C$ . Theorem 2 establishes which CDT graphs are  $(\vec{C}_g; \vec{P}_k)$ -UH digraphs. Elevating the resulting oriented cycles to the  $(k - 1)$ -power enables the construction, in Section 3, of fastened  $\mathcal{C}$ -UH graphs, with  $\mathcal{C}$  formed by copies of  $K_3$ ,  $K_4$ ,  $C_7$  and  $L(Q_3)$ , when  $\kappa = 3$ , via ‘zipping’ of the OO induced  $(k - 1)$ -arcs shared (as  $(k - 1)$ -paths) by pairs of OO  $g$ -cycles. In particular: **(a)** the Pappus (resp. Desargues) graph yields the disjoint union of two copies of the Menger graph of the self-dual  $(9_3)$ - (resp.  $(10_3)$ -) configuration, [6]; **(b)** the Coxeter graph yields the Klein graph on 56 vertices, [3, 17]; **(c)** the Biggs-Smith graph yields the Menger graph of a self-dual  $(102_4)$ -configuration, a non-line-graphical  $K_3$ -fastened  $\{K_4, L(Q_3)\}$ -UH graph, in contrasts with the self-dual  $(42_4)$ -configuration of [7], whose Menger graph is  $K_2$ -fastened  $\{K_4, K_{2,2,2}\}$ -UH.

## 2 $(C_g, P_k)$ -UH properties of CDT graphs

**Theorem 1** *Let  $G$  be a CDT graph of girth =  $g$  and  $AT = k$ . Then  $G$  is an SF  $(C_g; P_{i+2})$ -UH graph, for  $i = 0, 1, \dots, k - 2$ . In particular,  $G$  is a  $(C_g; P_k)$ -UH graph and has exactly  $2^{k-2}3ng^{-1}$   $g$ -cycles.*

*Proof.* We have to see that each CDT graph  $G$  with girth =  $g$  and  $AT = k$  is a  $(2^i - 1)$ -fastened  $(C_g; P_{k-i})$ -UH graph, for  $i = 0, 1, \dots, k - 2$ . In fact, each  $(k - i - 1)$ -path  $P = P_{k-i}$  of any such  $G$  is shared exactly by  $2^i$   $g$ -cycles of  $G$ , for  $i = 0, 1, \dots, k - 2$ . Moreover, each two of these  $2^i$   $g$ -cycles have just  $P$  in common. This and a simple counting argument for the number of  $g$ -cycles, as cited in the table above, yield the assertions in the statement.  $\square$

**Theorem 2** *The CDT graphs  $G$  of girth  $= g$  and  $AT = k$  that are not  $(\vec{C}_g; \vec{P}_k)$ -UH digraphs are the Petersen graph, the Heawood graph and the Foster graph. The remaining nine CDT graphs are fastened  $(\vec{C}_g; \vec{P}_k)$ -UH.*

*Proof.* Given a  $(\vec{C}_g; \vec{P}_k)$ -UH graph  $G$ , an assignment of an orientation to each  $g$ -cycle of  $G$  such that the two  $g$ -cycles shared by each  $(k-1)$ -path receive opposite orientations yields a  $(\vec{C}_g; \vec{P}_k)$ -orientation assignment (or  $(\vec{C}_g; \vec{P}_k)$ -OA). The collection of  $\eta$  oriented  $g$ -cycles corresponding to the  $\eta$   $g$ -cycles of  $G$ , for a particular  $(\vec{C}_g; \vec{P}_k)$ -OA will be called an  $(\eta\vec{C}_g; \vec{P}_k)$ -OAC.

The graph  $G = K_4$  on vertex set  $\{1, 2, 3, 0\}$  admits the  $(4\vec{C}_3; \vec{P}_2)$ -OAC

$$\{(123), (210), (301), (032)\}.$$

The graph  $G = K_{3,3}$  obtained from  $K_6$  (with vertex set  $\{1, 2, 3, 4, 5, 0\}$ ) by deleting the edges of the triangles  $(1, 3, 5)$  and  $(2, 4, 0)$  admits the  $(9\vec{C}_4; \vec{P}_3)$ -OAC

$$\{(1234), (3210), (4325), (1430), (2145), (0125), (5230), (0345), (5410)\}.$$

The graph  $G = Q_3$  with vertex set  $\{0, \dots, 7\}$  and edge set  $\{01, 23, 45, 67, 02, 13, 46, 57, 04, 15, 26, 37\}$  admits the  $(6\vec{C}_4; \vec{P}_2)$ -OAC

$$\{(0132), (1045), (3157), (2376), (0264), (4675)\}.$$

If  $G = Pet$  is the Petersen graph, then  $G$  can be obtained from the disjoint union of the 5-cycles  $\mu_\infty = (u_0u_1u_2u_3u_4)$  and  $\nu_\infty = (v_0v_2v_4v_1v_3)$  by the addition of the edges  $(u_x, v_x)$ , for  $x \in \mathbf{Z}_5$ . Apart from the two 5-cycles given above, the other ten 5-cycles of  $G$  can be denoted by  $\mu_x = (u_{x-1}u_xu_{x+1}v_{x+1}v_{x-1})$  and  $\nu_x = (v_{x-2}v_xv_{x+2}u_{x+2}u_{x-2})$ , for each  $x \in \mathbf{Z}_5$ . Then the following sequence of alternating 6-cycles and 2-arcs starts and ends up with opposite orientations:

$$\mu_2^-(u_3u_2u_1)\mu_\infty^+(u_0u_1u_2)\mu_1^-(u_2v_2v_0)\nu_0^-(v_3u_3u_2)\mu_2^+,$$

where the upper indices  $\pm$  indicate either a forward or backward selection of orientation and each 2-path is presented with the orientation of the previously cited 5-cycle but must be present in the next 5-cycle with its orientation reversed. Thus,  $Pet$  cannot be a  $(\vec{C}_5; \vec{P}_3)$ -UH digraph.

For each positive integer  $n$ , let  $I_n$  stand for the  $n$ -cycle  $(0, 1, \dots, n-1)$ , where  $0, 1, \dots, n-1$  are considered as vertices. If  $G = Hea$  is the Heawood graph, then  $G$  can be obtained from  $I_{14}$  by adding the edges  $(2x, 5+2x)$ , for  $x \in \{1, \dots, 7\}$  where operations are in  $\mathbf{Z}_{14}$ . The 28 6-cycles of  $G$  include the following 7 6-cycles:  $\gamma_x = (2x, 1+2x, 2+2x, 3+2x, 4+2x, 5+2x)$ , where  $x \in \mathbf{Z}_7$ . Then the following sequence of alternating 6-cycles and 3-arcs starts and ends up with opposite orientations for  $\gamma_0$ :

$$\gamma_0^+(2345)\gamma_1^-(7654)\gamma_2^+(6789)\gamma_3^-(ba98)\gamma_4^+(abcd)\gamma_5^-(10dc)\gamma_6^+(0123)\gamma_0^-,$$

(where tridecimal notation is used, up to  $d = 13$ ). Thus,  $Hea$  cannot be a  $(\vec{C}_7; \vec{P}_4)$ -UH digraph.

If  $G = Pap$  is the Pappus graph, then  $G$  can be obtained from  $I_{18}$  by adding the edges  $(1 + 6x, 6 + 6x)$ ,  $(2 + 6x, 9 + 6x)$ ,  $(4 + 6x, 11 + 6x)$ , for  $x \in \{0, 1, 2\}$ , where operations are mod 18. Then  $G$  admits a  $(18\vec{C}_6; \vec{P}_3)$ -OAC formed by the oriented 6-cycles  $A_0 = (123456)$ ,  $B_0 = (3210de)$ ,  $C_0 = (34bcde)$ ,  $D_0 = (0165gh)$ ,  $E_0 = (4329ab)$ , (where octodecimal notation is used, up to  $h = 17$ ), the 6-cycles  $A_x, B_x, C_x, D_x, E_x$  obtained by adding  $6x$  mod 18 to (the integer representations of) the vertices of  $A_0, B_0, C_0, D_0, E_0$ , where  $x \in \mathbf{Z}_3 \setminus \{0\}$ , and finally the 6-cycles  $F_0 = (23ef89)$ ,  $F_1 = (hg54ba)$ ,  $F_2 = (61idc7)$ .

If  $G = Dod$  is the dodecahedral graph, then  $G$  can be seen as a 2-covering graph of the Petersen graph  $H$ , where each vertex  $u_x$ , (resp.,  $v_x$ ), of  $H$  is covered by two vertices  $a_x, c_x$ , (resp.  $b_x, d_x$ ). This can be done so that a  $(12\vec{C}_5; \vec{P}_2)$ -OAC of  $G$  is formed by the oriented 5-cycles  $(a_0a_1a_2a_3a_4)$ ,  $(c_4c_3c_2c_1c_0)$  and for each  $x \in \mathbf{Z}_5$  also  $(a_xd_xb_{x-2}d_{x+1}a_{x+1})$  and  $(d_xb_{x+2}c_{x+2}c_{x-2}b_{x-2})$ .

If  $G = Des$  is the Desargues graph, then  $G$  can be obtained from the 20-cycle  $I_{20}$ , with vertices  $4x, 4x + 1, 4x + 2, 4x + 3$  redenoted alternatively  $x_0, x_1, x_2, x_3$ , respectively, for  $x \in \{0, \dots, 4\}$ , by adding the edges  $(x_3, (x + 2)_0)$  and  $(x_1, (x + 2)_2)$ , where operations are mod 5. Then  $G$  admits a  $(20\vec{C}_6; \vec{P}_3)$ -OAC formed by the oriented 6-cycles  $A_x, B_x, C_x, D_x$ , for  $x \in \{0, \dots, 4\}$ , where

$$\begin{aligned} A_x &= (x_0x_1x_2x_3(x+1)_0(x+4)_3), & B_x &= (x_1x_0(x+4)_3(x+4)_2(x+2)_1(x+2)_2), \\ C_x &= (x_2x_1x_0(x+3)_3(x+3)_2(x+3)_1), & D_x &= (x_0(x+4)_3(x+1)_0(x+1)_1(x+3)_2(x+3)_3). \end{aligned}$$

If  $G = Cox$  is the Coxeter graph, then  $G$  can be obtained from the three 7-cycles  $(u_1u_2u_3u_4u_5u_6u_0)$ ,  $(v_1v_3v_5v_0v_2v_4v_6)$ ,  $(t_1t_4t_0t_3t_6t_2t_5)$  by adding a copy of  $K_{1,3}$  with degree-3 vertex  $z_x$  and degree-1 vertices  $u_x, v_x, t_x$ , for each  $x \in \mathbf{Z}_7$ . Then  $Cox$  admits the  $(24\vec{C}_7; \vec{P}_3)$ -OAC

$$\begin{aligned} 0^1 &= (u_1u_2u_3u_4u_5u_6u_0), & 0^2 &= (v_1v_3v_5v_0v_2v_4v_6), & 0^3 &= (t_1t_4t_0t_3t_6t_2t_5), \\ 1^1 &= (u_1z_1v_1v_3z_3u_3u_2), & 1^2 &= (z_4v_4v_2v_0z_0t_0t_4), & 1^3 &= (t_6t_2t_5z_5u_5u_6z_6), \\ 2^1 &= (v_5z_5u_5u_4u_3z_3v_3), & 2^2 &= (t_6z_6v_6v_4v_2z_2t_2), & 2^3 &= (u_1z_1t_1t_4t_0z_0u_0), \\ 3^1 &= (v_5v_0z_0u_0u_6u_5z_5), & 3^2 &= (z_4t_4t_1z_1v_1v_6v_4), & 3^3 &= (t_6t_2z_2u_2u_3z_3t_3), \\ 4^1 &= (u_1u_0z_0v_0v_2z_2u_2), & 4^2 &= (t_6t_3z_3v_3v_1v_6z_6), & 4^3 &= (z_4u_4u_5z_5t_5t_1t_4), \\ 5^1 &= (z_4u_4u_3u_2z_2v_2v_4), & 5^2 &= (v_5v_3v_1z_1t_1t_5z_5), & 5^3 &= (t_6z_6u_6u_0z_0t_0t_3), \\ 6^1 &= (z_4v_4v_6z_6u_6u_5u_4), & 6^2 &= (v_5v_3z_3t_3t_0z_0v_0), & 6^3 &= (u_1u_2z_2t_2t_5t_1z_1), \\ 7^1 &= (u_1u_0u_6z_6v_6v_1z_1), & 7^2 &= (v_5z_5t_5t_2z_2v_2v_0), & 7^3 &= (z_4t_4t_0t_3z_3u_3u_4). \end{aligned}$$

If  $G = Tut$  is Tutte's 8-cage, then  $G$  can be obtained from  $I_{30}$ , with vertices  $5x, 5x + 1, 5x + 2, 5x + 3, 5x + 4, 5x + 5$  denoted alternatively  $x_0, x_1, x_2, x_3, x_4, x_5$ , respectively, for  $x \in \mathbf{Z}_5$ , by adding the edges  $(x_5, (x + 2)_0)$ ,  $(x_1, (x + 1)_4)$  and  $(x_2, (x + 2)_3)$ . Then  $G$  admits the  $(90\vec{C}_8; \vec{P}_5)$ -OAC formed by the oriented 8-cycles

$$\begin{aligned} A^0 &= (4_50_00_10_20_30_40_51_0), & B^0 &= (4_24_34_44_51_01_11_21_3), & C^0 &= (0_20_30_44_14_02_52_42_3), \\ D^0 &= (3_33_23_14_44_34_21_31_2), & E^0 &= (4_51_00_50_44_14_03_50_0), & F^0 &= (4_50_03_54_02_52_41_11_0), \\ G^0 &= (1_01_12_42_30_20_10_04_5), & H^0 &= (2_32_41_11_00_50_40_30_2), & I^0 &= (0_10_20_30_44_14_21_31_4), \\ J^0 &= (1_00_50_40_33_23_14_44_5), & K^0 &= (3_13_20_30_20_10_04_54_4), & L^0 &= (2_32_42_53_03_13_20_30_2), \\ M^0 &= (3_54_04_10_40_30_20_10_0), & N^0 &= (3_53_42_12_01_51_40_10_0), & O^0 &= (4_24_32_22_13_43_31_21_3), \\ P^0 &= (4_54_44_34_24_10_40_51_0), & Q^0 &= (4_04_14_21_31_41_53_02_5), & R^0 &= (0_10_20_33_23_13_01_51_4), \end{aligned}$$

together with those obtained from these 18 8-cycles by adding  $x \in \mathbf{Z}_5$  uniformly mod 5 to all subindices. Accordingly, these 8-cycles are denoted  $A^x, \dots, R^x$ , where  $x \in \mathbf{Z}_5$ .

If  $G = Fos$  is the Foster graph, then  $G$  can be obtained from  $I_{90}$ , with vertices  $5x, 5x + 1, 5x + 2, 5x + 3, 5x + 4, 5x + 5$  denoted alternatively  $x_0, x_1, x_2, x_3, x_4, x_5$ , respectively, for  $x \in \mathbf{Z}_{15}$ , by adding the edges  $(x_4, (x + 2)_1)$ ,  $(x_0, (x + 2)_5)$  and  $(x_2, (x + 6)_3)$ . The 90 10-cycles of  $G$  include the following 15 10-cycles, where  $x \in \mathbf{Z}_{15}$ .

$$\phi_x = (x_4 x_5 (x+1)_0 (x+1)_1 (x+1)_2 (x+1)_3 (x+1)_4 (x+1)_5 (x+2)_0 (x+2)_1),$$

Then the following sequence of alternating 10-cycles and 4-arcs:

$$\phi_0^+ [14] \phi_1^- [31] \phi_2^+ [34] \phi_3^- [51] \phi_4^+ [54] \phi_5^- [71] \phi_6^+ [74] \phi_7^- [91] \phi_8^+ [94] \phi_9^- [b_1] \phi_a^+ [b_4] \phi_b^- [d_1] \phi_c^+ [d_4] \phi_d^- [0_1] \phi_e^+ [0_4]$$

continues with  $\phi_0^-$ , of opposite orientation to that of the initial  $\phi_0^+$ , where  $[x_j]$  stands for a 3-path starting at the vertex  $x_j$  in the previously cited (to the left) oriented 10-cycle. Thus,  $Fos$  cannot be a  $(\vec{C}_{10}; \vec{P}_5)$ -UH digraph.

If  $G = BS$  is the Biggs-Smith graph, then  $G$  can be obtained from the four 17-cycles of vertices  $y_i$  with  $y \in \{A, B, C, D, E, F\}$  and heptadecimal subindices  $i$  (up to  $g = 16$ ):  $A = (A_0, A_1, \dots, A_g)$ ,  $D = (D_0, D_2, \dots, D_f)$ ,  $C = (C_0, C_4, \dots, C_d)$ ,  $F = (F_0, F_8, \dots, F_9)$  (advancing the subindices  $i$  in 1,2,4,8 units mod 17 stepwise from left to right, respectively) by adding a 6-vertex tree with degree-1 vertices  $A_i, C_i, D_i, F_i$  and degree-2 vertices  $B_i, E_i$  and containing the 3-paths  $A_i B_i C_i$  and  $D_i E_i F_i$ , for each  $i \in \mathbf{Z}_{17}$ .

$$\begin{aligned} S^0 &= (A_0 A_1 B_1 C_1 C_5 C_9 C_d C_0 B_0), & W^0 &= (A_0 A_1 B_1 E_1 F_1 F_9 F_0 E_0 B_0), \\ T^0 &= (C_0 C_4 B_4 A_4 A_3 A_2 A_1 A_0 B_0), & X^0 &= (C_0 C_4 B_4 E_4 D_4 D_2 D_0 E_0 B_0), \\ U^0 &= (E_0 F_0 F_9 F_1 F_a F_2 E_2 D_2 D_0), & Y^0 &= (E_0 B_0 A_0 A_1 A_2 B_2 E_2 D_2 D_0), \\ V^0 &= (E_0 D_0 D_2 D_4 D_6 D_8 E_8 F_8 F_0), & Z^0 &= (F_0 F_8 E_8 B_8 C_8 C_4 C_0 B_0 E_0), \end{aligned}$$

together with those obtained from these eight 9-cycles by adding  $x \in \mathbf{Z}_{17}$  uniformly mod 17 to all subindices. Accordingly, these 9-cycles are denoted  $T^x$ , etc., where  $x \in \mathbf{Z}_{17}$ .  $\square$

### 3 ‘Zipping’ the $(k - 1)$ -powers of $g$ -cycles ...

Given a CDT graph  $G$  with  $\kappa = 3$ , consider the collection  $\mathcal{C}_g^{k-1}(G)$  of  $(k - 1)$ -powers of oriented  $g$ -cycles in the  $(\eta\vec{C}_g; \vec{P}_k)$ -OAC of  $G$  in the proof of Theorem 2. If  $k = 3$ , then each arc  $\vec{e}$  of a member  $C^2$  of  $\mathcal{C}_g^2(G)$  is indicated by the middle vertex of the 2-arc  $\vec{E}$  in  $C$  for which  $\vec{e}$  stands, while the tail and head of  $\vec{e}$  are indicated by the tail and head of  $\vec{E}$ , respectively. This is the case in Subsections 3.1-2 below, in which we consider the CDT graphs  $G$  with  $\kappa = k = 3$  in order to ‘zip’ such  $C^2$ s along their OO arc pairs to obtain corresponding graphs  $Y(G)$  with  $\mathcal{C}$ -UH properties. In Subsection 3.3, we consider a similar construction for  $\kappa = 3 = k - 1$ , namely for the Biggs-Smith graph. In all these cases, the following sequence of operations is performed:

$$G \rightarrow (\eta\vec{C}_g; \vec{P}_k)\text{-OAC}(G) \rightarrow \mathcal{C}_g^{k-1}(G) \rightarrow Y(G).$$

The CDT graphs  $G$  with  $\kappa = 0$  do not admit the approach suggested in the previous paragraph for their  $g$ -cycles lack a  $(\vec{C}_g; \vec{P}_k)$ -OA; those with  $\kappa = 1$  admit the approach with  $Y(G) = G$  so nothing new is obtained more than a corresponding polyhedral graph (embeddable into the sphere) with faces delimited by  $g$ -cycles, namely the tetrahedral, 3-cube and dodecahedral graphs; those with  $\kappa = 2$  again admit the approach, but since  $\kappa/2 = k - 1$ , then  $Y(G) = (g - 1)G^{k-1}$ , the multigraph of multiplicity  $g - 1$  on the  $(k - 1)$ -th power of  $G$ .

### 3.1 ... for the Pappus, Desargues and $L(K_n)$ graphs, ...

If  $G$  is either the Pappus graph  $Pap$  or the Desargues graph  $Des$ , then  $\mathcal{C}_6^2(G)$  is formed by triangles conforming a graph  $Y(G)$  with just two connected components  $Y_1(G)$  and  $Y_2(G)$ .

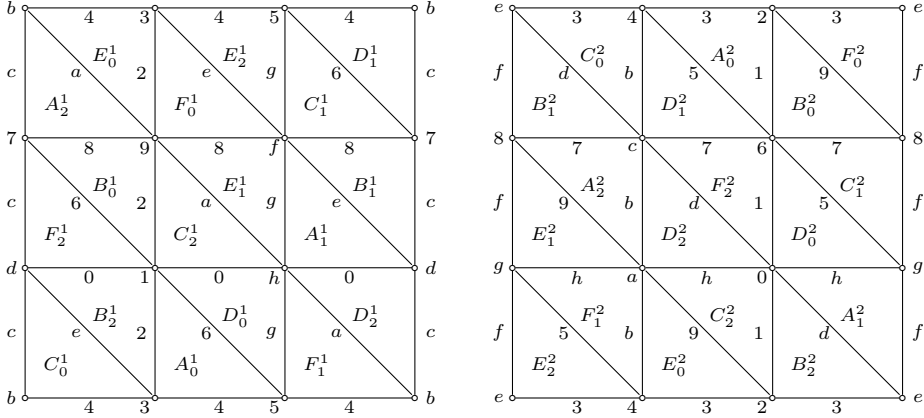


Figure 1: Toroidal cutouts of  $Y_1(Pap)$  and  $Y_2(Pap)$

Each of  $Y_1(Pap)$  and  $Y_2(Pap)$  is embeddable in a closed orientable surface  $T_1$  of genus 1, or 1-torus. In fact, Figure 1 shows toroidal cutouts of  $Y_1(Pap)$  and  $Y_2(Pap)$ . Notice that the copies of  $K_3$  in  $\mathcal{C}_6^2(G)$  are contractible in  $T_1$ . These triangles form two collections  $\mathcal{H}_1, \mathcal{H}_2$  of copies  $y_i^j$  of  $K_3$  closed under parallel translation, where  $y = A, B, C, D, E, F$ ;  $i = 0, 1, 2$  and  $j = 1, 2$ , namely: the nine of  $\mathcal{H}_1$  ( $\mathcal{H}_2$ ) with horizontal edge below (above) its opposite vertex. There is also a collection  $\mathcal{H}_0$  of nine non-contractible copies of  $K_3$  in  $G$ , traceable linearly in three different parallel directions, three such triangles per direction, with the edges of each triangle indicated by an associated common vertex of  $Pap$ .

There are embeddings of  $Y_1(Pap)$  and  $Y_2(Pap)$  in  $T_1$  for which  $\mathcal{H}_0$  and either  $\mathcal{H}_1$  or  $\mathcal{H}_2$  provide the composing faces. In addition, each of  $\mathcal{H}_1, \mathcal{H}_2$  and  $\mathcal{H}_0$  is formed by three classes of parallel elements, in the sense that any two of them in such a class do not have vertices in common. The self-dual (9<sub>3</sub>)-configuration in the following theorem is the Pappus configuration, [6].

**Theorem 3**  $Y_1(Pap)$  and  $Y_2(Pap)$  are isomorphic  $K_2$ -fastened  $\{H_0, H_1, H_2\}$ -

UH graphs, where  $H_i$  is a representative of  $\mathcal{H}_i$ , for  $i = 0, 1, 2$ . Moreover, each of  $Y_1(Pap)$  and  $Y_2(Pap)$  can be taken as the Menger graph of the Pappus self-dual  $(9_3)$ -configuration in 12 different forms, by selecting the point set  $\mathcal{P}$  and the line set  $\mathcal{L} \neq \mathcal{P}$  so that  $\{\mathcal{P}, \mathcal{L}\} \subset \{V(Pap), \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2\}$  and the incidence relation either as the inclusion of a vertex in a copy of  $K_3$  or as the containment by a copy of  $K_3$  of a vertex or as the sharing of an edge by two copies of  $K_3$ .

*Proof.* The statement can be established by managing the data given above. The 12 different claimed forms correspond to the arcs of the complete graph on vertex set  $\{V(Pap), \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2\}$ .  $\square$

If  $G = Des$ , then  $Y_1(G)$  and  $Y_2(G)$  are isomorphic  $K_2$ -fastened  $(K_4, K_3)$ -UH graphs, each formed by five copies of  $K_4$  and ten copies of  $K_3$ , with each such copy of  $K_3$ : **(a)** not forming part of a copy of  $K_4$  in  $Y_1(G)$  or  $Y_2(G)$ ; **(b)** having its edges indicated with a constant symbol, as shown in Figure 2.

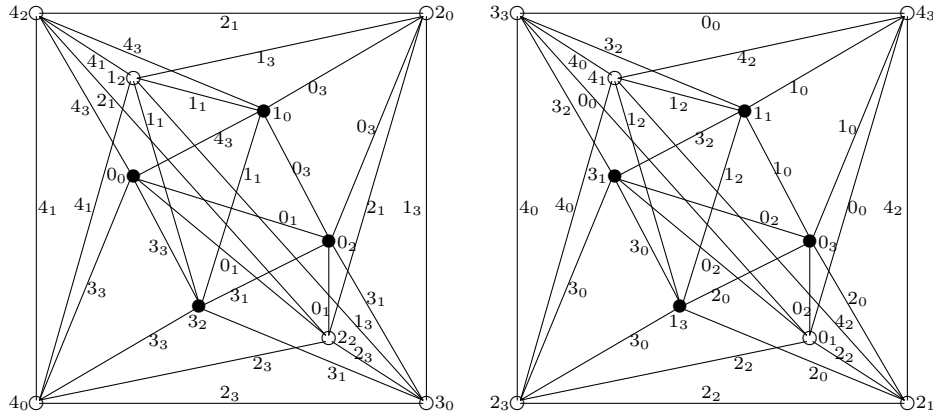


Figure 2: Representations of  $Y_1(Des)$  and  $Y_2(Des)$

Deleting a copy  $H$  of  $K_4$  from such  $Y_i(Des)$  yields a copy of  $K_{2,2,2}$ , four of whose composing copies of  $K_3$ , with no common edges, are faces of corresponding copies of  $K_4 \neq H$ ; the other four copies of  $K_3$  are among the ten mentioned copies of  $K_3$  in  $G$ . A realization of  $Y_1(G)$  (or  $Y_2(G)$ ) in 3-space can be obtained from a regular octahedron  $O_3$  realizing the  $K_{2,2,2}$  cited above via the midpoints of the four segments joining the barycenters of four edge-disjoint alternate triangles in  $O_3$  to the barycenter of  $O_3$  by constructing the tetrahedra determined by each of these alternate triangles and the nearest constructed midpoint, as well as the fifth central tetrahedron determined by the four midpoints.

By considering the barycenters of the resulting five tetrahedra and the segments joining them, a copy of  $K_5$  in 3-space is obtained. The geometric line graph  $L(K_5)$  it gives place to appears as a smaller version of  $Y_1(G)$  (or  $Y_2(G)$ ) contained in a octahedron  $O'_3 \subset O_3$ . This procedure may be repeated indefinitely, generating a sequence of realizations of  $Y_1(G)$  (or  $Y_2(G)$ ) in 3-space. Since

$Y_1(Des)$  and  $Y_2(Des)$  are isomorphic to  $L(K_5)$ , whose complement is  $Pet$ , then this sequence yields a corresponding sequence of realizations of  $Pet$  in 3-space.

We notice that the ten vertices and ten copies of  $K_3$  of either  $Y_i(Des)$  ( $i = 1, 2$ ) may be considered as the points and lines of the Desargues self-dual  $(10_3)$  configuration, and that the Menger graph of this coincides with  $Y_i(Des)$ , [6]. Each vertex of  $Y_i(Des)$  is the meeting vertex of two copies of  $K_4$  and three copies of  $K_3$  not forming part of a copy of  $K_4$ .

**Theorem 4**  $Y_1(Des)$  and  $Y_2(Des)$  are  $K_2$ -fastened  $\{K_4, K_3\}$ -UH graphs composed by five copies of  $K_4$  and ten copies of  $K_3$  each. Moreover, the ten vertices and ten copies of  $K_3$  in either graph constitute the Desargues self-dual  $(10_3)$  configuration, which has the graph itself as its Menger graph. Furthermore, both graphs are isomorphic to  $L(K_5)$ , whose complement is the Petersen graph.  $\square$

Theorem 4 can be partly generalized by replacing  $L(K_5)$  by  $L(K_n)$  ( $n \geq 4$ ). This produces a  $K_2$ -fastened  $\{K_{n-1}, K_3\}$ -UH graph.

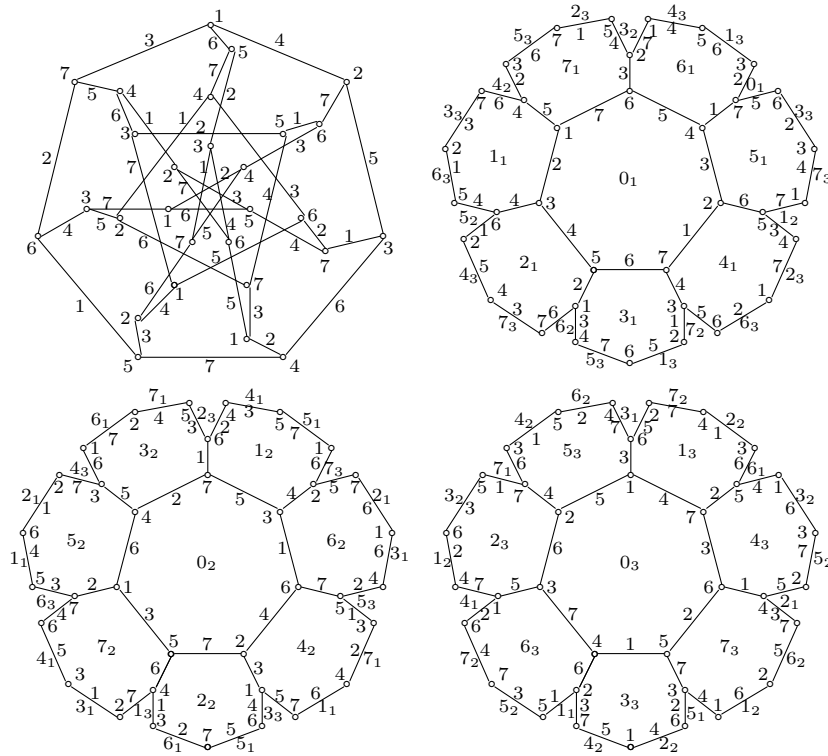


Figure 3:  $\mathcal{F}$ -colored  $Cox$  and the three charts of the Klein graph  $Y(Cox)$

s

**Theorem 5** The line graph  $L(K_n)$ , with  $n \geq 4$ , is a  $K_2$ -fastened  $\{K_{n-1}, K_3\}$ -UH graph with  $n$  copies of  $K_{n-1}$  and  $\binom{n}{3}$  copies of  $K_3$ .

*Proof.* We assume that each vertex of  $K_n$  is taken as a color of edges of  $L(K_n)$  under the following rule: Color all the edges between vertices of  $L(K_n)$  representing edges incident to a vertex  $v$  of  $K_n$  with color  $v$ . Then, each triple of edge colors for  $L(K_n)$  corresponds to the edges of a well determined copy of  $K_3$  in  $L(K_n)$ . Thus, there are exactly  $\binom{n}{3}$  copies of  $K_3$  intervening in  $L(K_n)$  looked upon as a  $\{K_{n-1}, K_3\}$ -UH graph.  $\square$

### 3.2 ... for the Coxeter graph ...

The Fano plane  $\mathcal{F}$ , with point set  $J_7 = \{1, \dots, 7\}$  and line set  $\{124, 235, 346, 457, 561, 672, 713\}$ , yields a coloring to the vertices and edges of  $G = Cox$ , represented on the upper left of Figure 3.

Observe that the colors of each vertex  $v$  of  $G$  and its three incident edges form a quadruple  $q$  whose complement  $\mathcal{F} \setminus q$  is a triangle of  $\mathcal{F}$  used as a ‘customary’ vertex denomination for  $v$ , [12], page 69. Then: **(a)** the triple formed by the colors of the edges incident to each vertex of  $G$  is a line of  $\mathcal{F}$ ; **(b)** the color of each edge  $e$  of  $G$  together with the colors of the endvertices of  $e$  form a line of  $\mathcal{F}$ . In this representation of  $G$ , the vertices  $u_x, z_x, v_x, t_x$  in the proof of Theorem 2 are depicted concentrically from the outside to the inside, respectively, starting with  $i = 1$ , say, on the upper middle vertices with ‘customary’ vertex denominations 257, 134, 356, 567.

The squares  $C^2$  of the 24 7-cycles  $C$  of  $G$  are taken with the following (cyclically presented) orientations, where each vertex  $v$  (resp. edge  $e$ ) of a  $C^2$  is indicated by the color of  $v$  (resp., subindicated by the color of the middle vertex of the 2-path of  $C$  that  $e$  represents). In fact, the resulting oriented 7-cycles can be denoted  $i^j$ , where  $i \in \{0\} \cup J_7$  and  $j \in J_3 = \{1, 2, 3\}$ , namely:

$$\begin{array}{lll}
0^1: & (1_2 3_4 5_6 7_1 2_3 4_5 6_7); & 0^2: & (1_3 5_7 2_4 6_1 3_5 7_2 4_6); & 0^3: & (1_5 2_6 3_7 4_1 5_2 6_3 7_4); \\
1^1: & (1_5 4_6 7_3 2_1 5_4 6_7 3_2); & 1^2: & (1_7 5_3 4_2 6_1 7_5 3_4 2_6); & 1^3: & (1_4 7_2 5_6 3_1 4_7 2_5 6_3); \\
2^1: & (1_2 5_4 3_7 6_1 2_5 4_3 7_6); & 2^2: & (1_3 2_7 5_6 4_1 3_2 7_5 6_4); & 2^3: & (1_5 3_6 2_4 7_1 5_3 6_2 4_7); \\
3^1: & (1_3 4_7 6_5 2_1 3_4 7_6 5_2); & 3^2: & (1_6 3_5 4_2 7_1 6_3 5_4 2_7); & 3^3: & (1_4 6_2 3_7 5_1 4_6 2_3 7_5); \\
4^1: & (1_7 4_3 5_6 2_1 7_4 3_5 6_2); & 4^2: & (1_5 7_6 4_2 3_1 5_7 6_4 2_3); & 4^3: & (1_4 5_2 7_3 6_1 4_5 2_7 3_6); \\
5^1: & (1_4 3_2 6_5 7_1 4_3 2_6 5_7); & 5^2: & (1_6 4_5 3_7 2_1 6_4 5_3 7_2); & 5^3: & (1_3 6_7 4_2 5_1 3_6 7_4 2_5); \\
6^1: & (1_7 2_3 6_5 4_1 7_2 3_6 5_4); & 6^2: & (1_6 7_5 2_4 3_1 6_7 5_2 4_3); & 6^3: & (1_2 6_4 7_3 5_1 2_6 4_7 3_5); \\
7^1: & (1_7 6_3 2_4 5_1 7_6 3_2 4_5); & 7^2: & (1_2 7_4 6_5 3_1 2_7 4_6 5_3); & 7^3: & (1_6 2_5 7_3 4_1 6_2 5_7 3_4).
\end{array}$$

Each 2-arc of  $G$  is cited exactly once in the oriented cycles  $i^j$ . Each 2-path of  $G$  appears twice in the  $i^j$ s, once for each one of its two 2-arcs. The assumed orientation of each  $C^2$  corresponds with the orientation of the corresponding 7-cycle  $C$ . Thus, we consider each 7-cycle  $C$  with the orientation it induces in the corresponding  $C^2$ , say  $i^j$ . In this case, we denote such a  $C$  as  $\underline{i}^j$ .

Each 2-path  $\underline{e}$  of  $G$  separates two of the 24 7-cycles of  $G$ , say  $\underline{i}^j$  and  $\underline{k}^\ell$ , with their orientations opposite over  $\underline{e}$ . Now,  $\underline{i}^j$  and  $\underline{k}^\ell$  restrict to the two different 2-arcs provided by  $\underline{e}$ , say 2-arcs  $e_1$  and  $e_2$ . Then,  $e_1$  and  $e_2$  represent corresponding arcs  $\underline{e}_1$  and  $\underline{e}_2$  in  $\underline{i}^j$  and  $\underline{k}^\ell$ , respectively.

Let us see that  $\underline{e}_1$  and  $\underline{e}_2$  can be ‘zipped’ into an edge  $e$  in the graph  $Y(G)$ . This, which happens to be the Klein graph, [3], graph 56B, can be assembled from the three charts shown on the upper right and bottom of Figure 3 by

‘zipping’ the 7-cycles  $i^j$ , interpreted all with counterclockwise orientation. Each of these three charts conforms a ‘rosette’, where the 7-cycles  $i^j$  with  $i \neq 0$  are represented as ‘petals’ of the ‘central’ 7-cycles  $0^1$ ,  $0^2$  and  $0^3$ . (The assemblage of  $Y(G)$  can be done also around the 7-cycles  $i^1$ ,  $i^2$  and  $i^3$ , for any  $i \in J_7$ , not just  $i = 0$ ). Moreover, each edge  $e$  in the external border of one of the three charts is accompanied by the denomination of a 7-cycle  $i_j$  incident to  $e$  which is a petal in one of the other two rosettes. Thus, the three charts can be assembled into the claimed graph  $Y(G)$ . Moreover, the 24 7-cycles  $i^j$  can be filled each with a corresponding 2-cell, so because of the cancelations of the two opposite arcs on each edge of  $Y(G)$  (for having opposite orientations makes them mutually cancelable),  $Y(G)$  becomes embedded into a closed orientable surface  $T_3$ . As for the genus of  $T_3$ , observe that

$$|V(Y(G))| = 2|V(X)| = 2 \times 28 = 56 \text{ and } |E(Y(G))| = 2|E(G)| = 2 \times 42 = 84,$$

so that by the Euler characteristic formula for  $T_3$  here,

$$|V(Y(G))| - |E(Y(G))| + |F_7(Y(G))| = 56 - 84 + 24 = -4 = 2 - 2.g(T),$$

and thus  $g = 3$ , so  $T_3$  is a 3-torus. This yields the Klein map (of Coxeter notation)  $\{7, 3\}_8$ , (see [17]; note: the Petrie polygons of this map are 8-cycles).

**Theorem 6** *The Klein graph  $Y(Cox)$  is a  $(C_7; P_2)$ -UH graph composed by 24 7-cycles that yield the Klein map  $\{7, 3\}_8$  in  $T_3$ .  $\square$*

For the Klein map  $\{7, 3\}_8$ , the 3-torus appeared originally dressed as the Klein quartic  $x^3y + y^3z + z^3x = 0$ , a Riemann surface and the most symmetrical curve of genus 3 over the complex numbers. The automorphism group for this Klein map is  $PSL(2, 7) = GL(3, 2)$ , the same group as for  $\mathcal{F}$ , whose index is 2 in the automorphism groups of  $Hea$ ,  $Cox$  and  $Y(Cox)$ .

**Corollary 7** *The graph  $Y'(Cox)$  whose vertices are the 7-cycles  $i_j$  of  $Y(Cox)$ , with adjacency between two vertices if their representative 7-cycles have an edge in common, is regular of degree 7, chromatic number 8 and has a natural triangular  $T_3$ -embedding yielding the dual Klein map  $\{3, 7\}_8$ .*

*Proof.* Each vertex  $i_j$  of  $Y'(Cox)$  is assigned color  $i \in \{0\} \cup J_7$ . Also, we have a partition of  $T_3$  into 24 connected regions, each region having exactly seven neighboring regions, with eight colors needed for a proper map coloring.  $\square$

### 3.3 ... and for the Biggs-Smith graph

The cubes  $C^3$  of the 136 9-cycles  $C$  of the Biggs-Smith graph  $G = BS$  are formed by three disjoint 3-cycles each, yielding a total of  $3 \times 136 = 408$  3-cycles. In fact, the  $(136 \vec{C}_9; \vec{P}_4)$ -OAC mentioned in the proof of Theorem 2 for  $G$  determines a  $(408 \vec{C}_3; \vec{P}_2)$ -OAC for  $Y(G)$ . The resulting oriented 3-cycles are ‘zipped’ along

the pairs of OO copies of  $\vec{P}_2$  obtained as cubes of OO copies of  $\vec{P}_4$  in  $G$ . The resulting ‘zipping’ of OO arcs yields 102 copies of  $K_4$ . These can be subdivided into six subcollections  $\{y^i\}$  of 17 copies each, where  $y \in \{A, B, C, D, E, F\}$  and  $i \in \{0, 1, \dots, 16 = g\} = \mathbf{Z}_{17}$ . The vertex sets  $V(y^i)$  of these copies of  $K_4$ , followed each by the corresponding set  $\Lambda(y_i)$  of copies of  $K_4$  containing the vertex  $y_i$ , are as follows:

$$\begin{aligned}
V(A^x) &= \{C_x, & D_x, & E_{x+4}, & E_{x-4}\}, & \Lambda(A_x) &= \{C^x, & D^x, & E^x, & E^{x+3}\}; \\
V(B^x) &= \{D_{x-8}, & D_{x-1}, & F_x, & F_{x+7}\}, & \Lambda(B_x) &= \{D^{x+2}, & D^{x-2}, & F^x, & F^{x-1}\}; \\
V(C^x) &= \{A_x, & F_x, & E_{x+2}, & E_{x-1}\}, & \Lambda(C_x) &= \{A^x, & F^{x+8}, & E^{x+4}, & E^{x-1}\}; \\
V(D^x) &= \{A_x, & D_x, & B_{x+2}, & B_{x-2}\}, & \Lambda(D_x) &= \{A^x, & D^x, & B^{x+2}, & B^{x+8}\}; \\
V(E^x) &= \{A_x, & A_{x-2}, & C_{x+1}, & C_{x-4}\}, & \Lambda(E_x) &= \{A^{x+4}, & A^{x-4}, & C^{x+1}, & C^{x-1}\}; \\
V(F^x) &= \{C_{x-8}, & F_{x-8}, & B_x, & B_{x+1}\}, & \Lambda(F_x) &= \{C^x, & F^{x+8}, & B^x, & B^{x-7}\};
\end{aligned}$$

where  $x \in \mathbf{Z}_{17}$ . This presentation emphasizes a duality property existing between the vertices of  $G$  and the copies of  $K_4$  in the cubes of the 9-cycles of  $G$ . A dual map realizing this duality property is given by an isomorphism from  $G$  (as presented in the proof of Theorem 2) onto a graph  $G'$  isomorphic to  $G$  and that can be obtained similarly from the four 17-cycles  $A' = (A^0, A^3, \dots, A^e)$ ,  $D' = (D^0, D^7, \dots, D^a)$ ,  $C' = (C^0, C^c, \dots, C^5)$ ,  $F' = (F^8, F^2, \dots, F^e)$  (which advance the vertex subindices in  $3 = 1 \times 3$ ,  $7 = 2 \times (-5)$ ,  $12 = 4 \times 3$ ,  $11 = 8 \times (-5)$  units mod 17 stepwise from left to right, respectively) by adding a 6-vertex tree with degree-1 vertices  $A^{3x}, C^{3x}, D^{-5x}, F^{8-5x}$  and degree-2 vertices  $B^{5-7x}, E^{10-6x}$  and containing the 3-paths  $A^{3i}B^{5-7i}C^{3i}$  and  $D^{-5x}E^{10-6x}F^{8-5x}$ , for each  $x \in \mathbf{Z}_{17}$ .

The cubes of the oriented cycles in the proof of Theorem 2 are:

$$\begin{aligned}
S^0 &\rightarrow \{E^0 \setminus A_e = (A_0, C_1, C_d), & E^4 \setminus A_4 = (A_1, C_5, C_0), & F^0 \setminus F_9 = (B_1, C_9, B_0)\}; \\
T^0 &\rightarrow \{E^4 \setminus C_5 = (C_0, A_4, A_1), & E^3 \setminus C_g = (C_4, A_3, A_0), & D^2 \setminus D_2 = (B_4, A_2, B_0)\}; \\
U^0 &\rightarrow \{C^1 \setminus A_1 = (E_0, F_1, E_2), & B^a \setminus D_8 = (F_0, F_a, D_2), & B^2 \setminus D_b = (F_9, F_2, D_0)\}; \\
V^0 &\rightarrow \{A^4 \setminus C_4 = (E_0, D_4, E_8), & B^8 \setminus F_f = (D_0, D_6, F_8), & B^a \setminus F_a = (D_2, D_8, F_0)\}; \\
W^0 &\rightarrow \{C^0 \setminus E_g = (A_0, E_1, F_0), & C^1 \setminus E_2 = (A_1, F_1, E_0), & F^0 \setminus C_9 = (B_1, F_9, B_0)\}; \\
X^0 &\rightarrow \{A^0 \setminus E_d = (C_0, E_4, D_0), & A^4 \setminus E_8 = (C_4, D_4, E_0), & D^2 \setminus A_2 = (B_4, D_2, B_0)\}; \\
Y^0 &\rightarrow \{C^1 \setminus F_1 = (E_0, A_1, E_2), & D^2 \setminus B_4 = (B_0, A_2, D_2), & D^0 \setminus B_f = (A_0, B_2, D_0)\}; \\
Z^0 &\rightarrow \{F^8 \setminus B_9 = (F_0, B_8, C_0), & F^9 \setminus B_g = (F_8, C_8, B_0), & A^4 \setminus D_4 = (E_8, C_4, E_0)\}.
\end{aligned}$$

Because of the properties of  $G$ , it can be seen that  $Y(G)$  is a  $K_4$ -UH graph. Moreover, the vertices and copies of  $K_4$  in  $Y(G)$  are the points and lines of a self-dual  $(102_4)$ -configuration which in turn has  $Y(G)$  as its Menger graph. (Compare with [6, 7]). However, in view of Beineke’s characterization of line graphs [1], and observing that  $Y(G)$  contains induced copies of  $K_{1,3}$ , which are forbidden for line graphs of simple graphs, we conclude that  $Y(G)$  is non-line-graphical, as commented above for the graph treated in [7], the Menger graph of a self-dual  $(42_4)$ -configuration.

**Theorem 8**  *$Y(BS)$  is an edge-disjoint union of 102 copies of  $K_4$  with four such copies incident to each vertex. Moreover,  $Y(BS)$  is a non-line-graphical  $K_4$ -UH graph. Its vertices and copies of  $K_4$  are the points and lines, respectively, of a self-dual  $(102_4)$ -configuration, which in turn has  $Y(BS)$  as its Menger graph. This is an arc-transitive graph with regular degree 12, diameter 3, distance distribution  $(1, 12, 78, 11)$  and automorphism-group order 2448. Its associated Levi*

graph is a 2-arc-transitive graph with regular degree 4, diameter 6, distance distribution  $(1, 4, 12, 36, 78, 62, 11)$  and automorphism-group order 4896.  $\square$

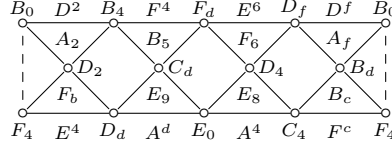


Figure 4: Copy  $\overline{A^0}$  of  $L(Q_3)$  in  $Y(BS)$

In the realization of the subgraph of  $G$  that yields any one of the 102 copies of  $K_4$  (produced by taking the cubes of the 102 9-cycles) there are four degree-3 vertices (exactly those described in the data given above) and twelve degree-2 vertices (exactly the two middle vertices in each path of length 3 realizing an edge of such a copy of  $K_4$ ). These twelve vertices form a copy  $\mathcal{L}$  of  $L(Q_3)$  in  $Y(G)$ . For the copy  $A^0$  of  $K_4$  in  $Y(G)$ , the copy  $\overline{A^0}$  of  $L(Q_3)$  in  $Y(G)$  accompanying such  $\mathcal{L}$  can be represented as in Figure 4, where: **(a)** the leftmost and rightmost dotted lines are to be identified by parallel translation; **(b)** each one of the eight copies of  $K_3$  forms part of a corresponding copy of  $K_4$  (among the 102 of  $Y(G)$ ) cited by name about its horizontal edge. By presenting the elements of the shown representation orderly, we may indicate the copies  $y^0$  of  $L(Q_3)$  as follows, for  $y = A, B, C, D, E, F$ :

$$\begin{aligned} \overline{A^0}: & (B_0 B_4 F_d D_f) (F_4 D_d E_0 C_4) D_2 C_d D_4 B_d (D^2 A_2 F^4 A_5 B^6 F_6 D^f A_f) (B^4 F_b A^d E_9 A^4 E_8 F^c B_c) \\ \overline{B^0}: & (D_0 F_8 E_7 D_b) (D_7 D_d E_0 F_9) F_f E_9 E_f F_9 (B^8 D_6 C^8 A_8 A^b C_b B^2 F_2) (B^f F_5 A^d C_d C^9 A_9 B^9 D_1) \\ \overline{C^0}: & (A_1 B_9 F_8 F_1) (F_9 F_9 B_1 A_9) D_1 B_0 D_9 E_0 (D^1 B_3 F^9 C_8 B^1 D_a C^1 E_2) (B^9 D_7 F^0 C_9 D^9 B_c C^9 E_f) \\ \overline{D^0}: & (A_1 E_2 D_f A_f) (E_f A_g A_2 D_2) E_0 C_f B_0 C_2 (C^1 F_1 A^f E_b D^f B_d E^1 C_e) (C^9 F_9 E^2 C_3 D^2 B_4 A^2 E_6) \\ \overline{E^0}: & (A_1 C_5 B_d A_f) (A_d A_g B_1 C_9) C_0 B_c B_0 C_c (E^4 A_4 F^d F_5 D^f D_f E^1 C_2) (E^9 C_c D^9 D_g F^0 F_9 E^d A_a) \\ \overline{F^0}: & (A_0 C_1 B_9 F_0) (E_9 E_0 A_1 C_5) C_d F_1 C_0 E_1 (E^0 A_e F^9 B_a F^8 B_8 C^0 E_9) (A^d D_d C^1 E_2 E^4 A_4 A^5 D_5) \end{aligned}$$

and obtain the remaining  $\overline{y^i}$  by uniform translations mod 17, for any  $i \in \mathbf{Z}_{17}$ .

Each vertex of  $Y(G)$  belongs exactly to twelve such  $\mathcal{L}$ s. Figure 5 shows the complements of  $A_0$  in four of the twelve copies of  $L(Q_3)$  containing it (sharing the long vertical edges), where the black vertices form the 4-cycles containing  $A_0$ , and some edges and vertices appear repeated twice, once per copy of  $L(Q_3)$ ; for example, the leftmost and rightmost edges must be identified by parallel translation; alternate internal anti-diagonal 2-paths in the figure also coincide with their directions reversed; (notice that the middle vertices of these 2-paths are the neighbors of  $A_0$  in  $G$ , and that their degree-1 vertices are at distance 2 from  $A_0$ , again in  $G$ ). The oriented 9-cycles of the  $(\eta\vec{C}_9; \vec{P}_4)$ -OAC of  $G$  cited in the proof of Theorem 2 intervene, as indicated on the figure, in the formation of the oriented 3-cycles and copies of  $L(Q_3)$  induced by the long vertical edges  $(C_d C_1, A_3 C_4, D_0 B_2, E_1 F_0)$  resp. for  $E^0, E^3, D^0, C^0$  and the 6-cycles they separate  $((C_d B_5 A_3 C_4 A_4 C_1), (A_3 E_4 D_0 B_2 D_4 C_4), (D_0 F_2 E_1 F_0 F_a B_2), (E_1 E_9 C_d C_1 B_9 F_0))$  resp. for  $\overline{E^4}, \overline{D^2}, \overline{C^1}, \overline{F^0}$ .

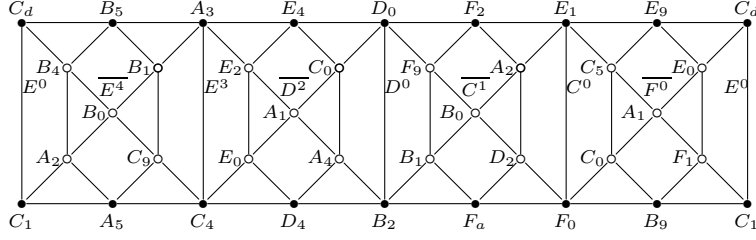


Figure 5: Complements of  $A_0$  in four of the twelve copies of  $L(Q_3)$

The information in the figure can be set as in the following arrangement, followed by two additional arrangements that complete all the information provided by the copies of  $K_4$  and  $L(Q_3)$  that contain  $A_0$ :

$E^0$	$\overline{E^4}$	$E^3$	$\overline{D^2}$	$D^0$	$\overline{C^1}$	$C^0$	$\overline{F^0}$								
$C_d$	$B_4$	$B_0$	$B_1$	$A_3$	$E_2$	$A_1$	$C_0$	$D_0$	$F_9$	$B_0$	$A_2$	$C_5$	$A_1$	$E_0$	
$C_1$	$A_2$	$A_5$	$C_9$	$C_4$	$E_0$	$D_4$	$A_4$	$B_2$	$B_1$	$F_a$	$D_2$	$F_0$	$C_0$	$B_9$	$F_1$
$E^0$	$\overline{D^g}$	$C^0$	$\overline{D^1}$	$E^3$	$\overline{E^2}$	$D^0$	$\overline{E^1}$								
$C_1$	$D_1$	$A_1$	$A_f$	$E_g$	$C_3$	$A_g$	$B_1$	$A_3$	$C_2$	$A_1$	$B_g$	$B_f$	$B_e$	$A_g$	$A_2$
$A_e$	$B_g$	$D_e$	$C_e$	$E_1$	$A_2$	$E_3$	$D_g$	$C_g$	$A_f$	$C_b$	$B_3$	$B_2$	$B_1$	$C_a$	$C_f$
$E^0$	$\overline{D^f}$	$D^0$	$\overline{C^g}$	$C^0$	$\overline{F^g}$	$E^3$	$\overline{E^g}$								
$A_e$	$E_f$	$A_g$	$C_0$	$D_0$	$F_8$	$B_0$	$A_f$	$E_g$	$C_c$	$A_g$	$E_0$	$C_4$	$B_d$	$B_0$	$B_g$
$C_d$	$E_0$	$D_d$	$A_d$	$B_f$	$B_g$	$F_7$	$D_f$	$F_0$	$C_0$	$B_8$	$F_g$	$C_g$	$A_f$	$A_c$	$C_8$

(Some edges are shared by two different of these three arrangements. In fact, each of the edges bordering the 2-paths mentioned above in anti-diagonal 4-paths is present also in the second or third arrangement. For example, the edge  $B_1A_3$  of  $E^4$  on the figure appears in the second arrangement).

In the same way, the vertices  $B_0$ ,  $C_0$ ,  $D_0$ ,  $E_0$  and  $F_0$  admit similar arrangements (see the Appendix) and additions mod 17 yield the remaining information for neighboring copies of  $K_4$  and  $L(Q_3)$  at each vertex of  $Y(G)$ .

**Theorem 9** *The graph  $Y(BS)$  is an SF  $(L(Q_3); K_3)$ -UH graph. Moreover, each two copies of  $L(Q_3)$  sharing a copy  $H$  of  $K_3$  in  $Y(BS)$  also share  $H$  with exactly one copy of  $K_4$  in  $Y(BS)$ . Furthermore, each 4-cycle of  $Y(BS)$  exists in just one copy of  $L(Q_3)$  in  $G$ . Thus,  $Y(BS)$  is  $K_3$ -fastened  $\{K_4, L(Q_3)\}$ -UH.*

*Proof.* Given a copy  $H$  of  $L(Q_3)$  in  $Y(BS)$  and a copy  $\Delta$  of  $K_3$  in  $H$ , there exists a unique copy  $\neq H$  of  $L(Q_3)$  that shares with  $H$  the subgraph  $\Delta$ . In addition, any edge of  $H$  is shared by exactly three other copies  $H' \neq H$ ,  $H'' \neq H$  and  $H''' \neq H$  of  $L(Q_3)$ . Because of the symmetry reigning between the copies of  $K_4$  and of  $L(Q_3)$  in  $Y(BS)$ , the statement follows.  $\square$

### 3.4 Appendix: local behavior of $Y(BS)$

Here is the data that must replace the symbols in Figure 5 in order to yield the complements of  $y_0$  in the copies of  $K_4$  and  $L(Q_3)$  incident to  $y_0$ , for  $y = A, B, C, D, E, F$ , where the rows are cited in parentheses that are preceded by  $y_0$  and the fourth rows cite each appearing vertex just once:

$(E^0 \overline{E^4 E^3 \overline{D^2 D^0 \overline{C^1 C^0 \overline{F^0}}}})$ $(C_d B_5 A_3 E_4 D_0 F_2 E_1 E_9)$ $(B_4 B_1 E_2 C_0 F_9 A_2 C_5 E_0)$ $A_0(B_0 A_1)$ $(A_2 C_9 E_0 A_4 B_1 D_2 C_0 F_1)$ $(C_1 A_5 C_4 C_7 B_2 F_a F_0 B_9)$	$(E^0 \overline{D^9 C^0 \overline{D^1 E^3 \overline{E^2 D^0 \overline{E^1}}}})$ $(C_1 E_e E_g D_3 A_3 C_7 B_f C_6)$ $(D_1 A_f C_3 B_1 C_2 B_g B_e A_2)$ $A_0(A_1 A_g)$ $(B_g C_e A_2 D_g A_f B_3 B_1 C_f)$ $(A_e D_e E_1 E_3 C_g C_b B_2 C_a)$	$(E^0 \overline{D^f D^0 \overline{C^9 C^0 \overline{F^9 E^3 \overline{E^9}}}})$ $(A_e E_d D_0 F_f E_g E_8 C_4 B_c)$ $(E_f C_0 F_8 A_f C_c E_0 B_d B_g)$ $A_0(A_g B_0)$ $(E_0 A_d B_g D_f C_0 F_g A_f C_8)$ $(C_d D_d B_f F_7 F_0 B_8 C_g A_c)$
$(F^0 \overline{F^8 F^9 \overline{A^4 D^2 \overline{A^0 D^f \overline{A^d}}}})$ $(F_9 A_9 C_8 D_8 D_2 F_4 B_d D_b)$ $(E_8 C_d E_4 F_0 D_d C_4 B_9 D_0)$ $B_0(E_0 C_0)$ $(C_4 E_9 D_0 B_8 C_d D_4 F_0 E_d)$ $(C_9 A_8 F_8 D_6 B_4 F_d D_f D_9)$	$(F^0 \overline{E^0 D^f \overline{E^9 F^9 \overline{E^3 D^2 \overline{E^4}}}})$ $(C_9 C_e A_f A_c C_8 C_3 A_2 A_5)$ $(A_d A_1 C_g C_d A_4 A_g C_1 C_4)$ $B_0(C_0 A_0)$ $(A_g C_5 C_4 A_e A_1 C_c C_d A_3)$ $(B_1 B_e B_d B_c B_g B_3 B_4 B_5)$	$(F^0 \overline{C^1 D^2 \overline{D^0 D^f \overline{C^9 F^9 \overline{C^0}}}})$ $(B_1 F_a D_2 C_2 A_f F_f F_8 D_g)$ $(B_2 F_0 E_f A_1 E_g D_0 F_1 A_g)$ $B_0(A_0 E_0)$ $(D_0 E_1 A_g E_2 F_0 B_f A_1 F_g)$ $(F_9 F_2 A_2 C_f D_f F_7 B_g D_1)$
$(A^0 \overline{A^4 F^8 \overline{F^9 E^9 \overline{E^3 E^4 \overline{D^2}}}})$ $(D_0 D_6 B_8 F_g A_g C_3 A_4 B_2)$ $(F_8 B_4 C_g E_0 A_2 C_8 D_4 A_0)$ $C_0(B_0 C_4)$ $(C_8 D_2 A_0 E_8 B_4 B_g E_0 A_3)$ $(E_4 D_8 F_0 E_g C_c B_3 A_1 E_2)$	$(A^0 \overline{F^4 E^4 \overline{E^8 F^8 \overline{E^3 E^9 \overline{E^4}}}})$ $(E_4 F_5 C_5 A_6 B_8 C_3 A_d E_c)$ $(A_5 B_d A_9 B_4 B_c C_9 F_d C_8)$ $C_0(C_4 C_d)$ $(C_9 F_4 C_8 B_5 B_d A_8 B_4 A_c)$ $(E_d E_5 A_4 A_7 B_9 B_3 C_c F_c)$	$(A^0 \overline{D^f E^9 \overline{E^0 E^4 \overline{F^0 F^8 \overline{A^d}}}})$ $(E_d E_f A_g B_e C_5 F_f F_0 D_9)$ $(A_e E_0 B_1 B_d E_9 A_0 D_f C_9)$ $C_0(C_d B_0)$ $(A_0 D_d C_9 A_f E_0 C_1 B_d F_9)$ $(D_0 B_f A_d C_e A_1 F_7 B_9 D_b)$
$(D^0 \overline{D^2 A^0 \overline{C^1 B^8 \overline{B^2 C^1}}})$ $(A_0 A_3 E_4 F_2 F_8 D_a F_2 E_1)$ $(C_4 E_2 D_8 B_0 F_1 D_4 A_2 F_0)$ $D_0(E_0 D_2)$ $(D_4 A_1 F_0 B_4 E_2 E_8 B_0 F_a)$ $(B_2 A_4 C_0 F_a D_6 E_a F_9 B_1)$	$(D^0 \overline{A^f B^2 \overline{B^4 A^0 \overline{B^6 B^8 \overline{A^2}}}})$ $(B_2 C_b D_b F_3 E_4 F_5 F_f B_6)$ $(F_b E_f F_d E_2 E_6 D_d C_f D_4)$ $D_0(D_2 D_f)$ $(D_d C_2 D_4 E_b E_f F_4 E_2 F_6)$ $(B_f B_b F_2 F_c E_d F_e D_6 C_6)$	$(D^0 \overline{C^9 B^8 \overline{B^0 B^2 \overline{F^9 A^0 \overline{D^f}}}})$ $(B_f B_g F_8 E_7 D_b B_9 C_0 A_d)$ $(F_7 B_0 E_9 E_f B_d F_0 A_g D_d)$ $D_0(D_f E_0)$ $(F_0 A_f D_d F_g B_0 D_9 E_f C_d)$ $(A_0 E_g F_f D_7 F_9 C_9 E_d A_e)$
$(C^1 \overline{F^1 C^9 \overline{D^f A^d \overline{A^0 A^4 \overline{C^2}}}})$ $(A_1 C_2 E_f A_e C_d F_d D_4 A_4)$ $(A_f D_2 E_d A_0 B_4 D_f B_2 C_0)$ $E_0(B_0 D_0)$ $(D_f A_2 C_0 B_f D_2 B_d A_0 E_4)$ $(E_2 C_f A_g A_d D_d F_4 C_4 A_3)$	$(C^1 \overline{B^a A^4 \overline{B^8 C^9 \overline{A^d B^2 \overline{C^1}}}})$ $(E_2 E_a E_8 E_6 E_f E_7 E_9 E_b)$ $(D_6 F_9 F_7 D_2 D_b F_8 F_a D_f)$ $E_0(D_0 F_0)$ $(F_8 F_2 D_f D_8 F_9 F_f D_2 D_9)$ $(F_1 D_a D_4 F_6 F_g D_7 D_d F_b)$	$(C^1 \overline{F^0 A^d \overline{C^9 A^4 \overline{F^9 C^9 \overline{C^0}}}})$ $(F_1 C_1 C_d A_9 E_8 C_c A_g D_g)$ $(B_9 A_0 C_8 F_9 E_g C_0 B_1 F_8)$ $E_0(F_0 B_0)$ $(C_0 E_1 F_8 C_9 A_0 B_8 F_9 B_g)$ $(A_1 C_5 E_9 A_8 C_4 C_g F_g D_1)$
$(C^0 \overline{F^0 F^8 \overline{A^d B^0 \overline{B^2 B^a \overline{C^0}}}})$ $(A_0 C_1 B_9 B_d D_f E_b F_a B_2)$ $(C_d F_1 D_b B_0 E_2 E_9 B_1 D_0)$ $F_0(E_0 F_9)$ $(E_9 A_1 D_0 C_9 F_1 D_d B_0 F_2)$ $(E_1 D_e D_8 B_a B_9 A_7 F_7 D_5)$	$(C^0 \overline{B^1 B^a \overline{C^9 F^8 \overline{C^8 B^0 \overline{B^9}}}})$ $(E_1 D_e D_8 B_a B_9 A_7 F_7 D_5)$ $(E_a F_g A_8 F_1 D_7 E_8 D_g E_9)$ $F_0(F_9 F_8)$ $(E_8 D_1 E_9 D_a F_g A_9 F_1 E_7)$ $(E_g D_c F_a A_a B_8 B_7 D_9 D_3)$	$(C^0 \overline{C^9 B^0 \overline{F^9 B^a \overline{C^9 F^8 \overline{F^9}}}})$ $(E_g A_f D_f F_6 D_8 E_4 C_0 C_c)$ $(F_f B_0 D_4 F_g C_8 D_0 A_g E_8)$ $F_0(F_8 E_0)$ $(D_0 B_g E_8 E_f B_0 D_6 F_g C_4)$ $(A_0 B_f F_7 E_6 D_2 B_4 B_8 C_g)$

## References

- [1] L. W. Beineke, *Derived graphs and digraphs*, in Beiträge zum Graphentheorie, Teubner (1968) 17–33
- [2] N. L. Biggs and D. H. Smith, *On trivalent graphs*, Bull. London Math. Soc., **3**(1971), 155–158.

- [3] I. Z. Bouwer et al., *The Foster Census*, R. M. Foster's Census of Connected Symmetric Trivalent Graphs, Charles Babbage Res. Ctr., Canada 1988.
- [4] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, New York 1989.
- [5] G. L. Cherlin, *The Classification of Countable Homogeneous Directed Graphs and Countable Homogeneous  $n$ -tournaments*, *Memoirs Amer. Math. Soc.*, vol. 131, number 612, Providence RI, January 1988.
- [6] H. S. M. Coxeter, *Self-dual configurations and regular graphs*, *Bull. Amer. Math. Soc.*, **56**(1950), 413–455.
- [7] I. J. Dejter, *On a  $\{K_4, K_{2,2,2}\}$ -ultrahomogeneous graph*, *Australasian Journal of Combinatorics*, **44**(2009), 63–76.
- [8] I. J. Dejter, *On a  $\vec{C}_4$ -ultrahomogeneous oriented graph*, preprint 2009.
- [9] R. Fraïssé, *Sur l'extension aux relations de quelques propriétés des ordres*, *Ann. Sci. École Norm. Sup.* 71 (1954), 363–388.
- [10] Frucht, R., *A canonical representation of trivalent hamiltonian graphs*, *J. Graph Th.*, bf 1(1976), 45–60.
- [11] A. Gardiner, *Homogeneous graphs*, *J. Combinatorial Theory (B)*, **20** (1976), 94–102.
- [12] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer-Verlag, 2001.
- [13] D. C. Isaksen, C. Jankowski and S. Proctor, *On  $K_*$ -ultrahomogeneous graphs*, *Ars Combinatoria*, Volume LXXXII, (2007), 83–96.
- [14] A. H. Lachlan and R. Woodrow, *Countable ultrahomogeneous undirected graphs*, *Trans. Amer. Math. Soc.* 262 (1980), 51–94.
- [15] C. Ronse, *On homogeneous graphs*, *J. London Math. Soc. (2)* **17** (1978), 375–379.
- [16] J. Sheehan, *Smoothly embeddable subgraphs*, *J. London Math. Soc. (2)* **9** (1974), 212–218.
- [17] E. Schulte and J. M. Wills, *A Polyhedral Realization of Felix Klein's Map  $\{3,7\}_8$  on a Riemann Surface of Genus 3*, *J. London Math. Soc.*, Second Ser. **32** (1985), 539–547.