Equitable Factorizations of Hamming Shells

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Abstract

We construct a 1-factorization of the complement $\Sigma_m$ of the linear Hamming code of length $m = m_r = 2^r - 1$ in the $m$-cube $Q_m$, for $r \geq 2$, having the following equitable property: its component 1-factors intersect each Cayley parallel 1-factor of $Q_m$ at a constant number of edges, (namely $2^{m_r - r - 1}$ edges). In the way to that construction, we find an equitable $m_{r-1}$-factorization of $\Sigma_m$ formed by two factors $\Omega_r, \Omega'_r$, specifically two spanning regular subgraphs, self-complementary in $\Sigma_m$. These results were already known for $r \leq 3$, where $\Omega_3$ and $\Omega'_3$ coincide with the so-called Foster graph.

1 Introduction

Let $1 < m \in \mathbb{Z}$. We define the $m$-cube $Q_m$ as the undirected Cayley graph $\text{Cay}(\mathbb{Z}_2^m, B_m)$ of the Abelian group $\mathbb{Z}_2^m$ with respect to the generating set $B_m$ given by the standard basis of $\mathbb{Z}_2^m$:

$$B_m = \{e_1^m = 10 \ldots 0, e_2^m = 01 \ldots 0, \ldots, e_m^m = 00 \ldots 1\} \subset \mathbb{Z}_2^m.$$

As a result, the vertex set $V = V(Q_m)$ equals $\mathbb{Z}_2^m = (GF_2)^m = \{0, 1\}^m$ and the edge set $E(Q_m)$ admits the Cayley 1-factorization $\{f_1^m, \ldots, f_m^m\}$, that is $E(Q_m)$ equals the disjoint union $f_1^m \cup \ldots \cup f_m^m$, with

$$f_m^\ell = \{(v, w) \in V^2 : v - w = e_m^\ell\},$$

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the Cayley parallel 1-factor formed by the edges labelled by $e_\ell^m \in B_m$, i.e. running along the $\ell$-th coordinate direction, for $\ell = 1, \ldots, m$.

A perfect 1-error-correcting code $C$ of length $m$, ([?]), or efficient dominating set $C$ of $Q_m$, ([?]), is a maximum stable set of $Q_m$ with minimum distance 3, so that $d(v, C) = 1$ for every $v \in V \setminus C$. It is known that it exists if and only if $m = 2^r - 1$, for $1 \leq r \in \mathbb{Z}$, in which case we denote $m = m_r$. We write $C = \mathcal{H}_{m_r} = \mathcal{H}_m$ if $C$ is a linear subspace, i.e. the Hamming code of length $m_r$ that behaves as a subgroup of the Abelian group $V(Q_m) = \mathbb{Z}_2^m$. Recall that $\mathcal{H}_m$ is the kernel of the parity check $m \times r$-matrix $H_r$ ([?], pg. 49) whose $\ell$-th row is the $r$-tuple $\lambda_\ell \in \mathbb{Z}_2^r \setminus \{00\ldots0\} = \mathbb{Z}_2^r \setminus \{0\}$ associated to the binary expression of the integer $\ell$, (with enough zeros to the left in order to attain length $r$), for $\ell \in \{1, \ldots, m\}$, ([?], pgs. 111-112, 121-124). The complement $Q_m \setminus \mathcal{H}_m$ of this Hamming kernel in $Q_m$ is said to be the Hamming shell $\Sigma_m$. (In fact, $Q_m$ may be considered as a ‘seed’ separable into its ‘kernel’, $\mathcal{H}_m$, and its ‘shell’, $\Sigma_m$). Notice that the triples of linearly dependent rows of $H_r$ form a linear Steiner triple system $L_r$ that yields, via edges along the Cayley parallel 1-factors $f_\ell^m$, the unique way to connect vertices of $\mathcal{H}_m$ at distance 3, via shortest paths.

The mentioned facts imply that both $Q_m$ and $\Sigma_m$ are bipartite, regular and vertex- and edge-transitive graphs, with $\Sigma_m$ of codegree 1 in $Q_m$. This inspired our present aim of constructing, for each $r \geq 2$ and $m = m_r = 2^r - 1$, a 1-factorization $\mathcal{F}$ of $\Sigma_m$ with its component 1-factors intersecting each $f_\ell^m$ at a constant number of edges, (namely $2^{m_r-r-1}$ edges), thus forming an equitable factorization of $\Sigma_m \subset Q_m$. (This is trivially seen for $r = 2$ and exemplified adequately for $r = 3$ and $r = 4$ in the two following sections).

In the way to that construction, we find an equitable $(\frac{m_r-1}{2})$- or $m_{r-1}$-factorization of $\Sigma_m$ formed by two factors, namely two spanning regular subgraphs of $\Sigma_m$, (which for $r = 3$ are isomorphic to the so-called Foster graph, an edge- but not vertex-transitive graph, [?]). These subgraphs are self-complementary in $\Sigma_m$. For $r = 3$, these are results of [?, ?, ?].

2 Preliminary Notions

A graph map $\pi : G \rightarrow H$ is an $h$-covering graph map with source $G$ and target $H$ if the inverse image via $\pi$ of each vertex or edge of $H$ has cardinality $h$ in $G$. Then we say that $G$ is a covering graph of $H$, that $H$ is a quotient graph of $G$, that $G$ covers $H$ and that each vertex $v$ of $H$ is the target of $\pi^{-1}(v)$. 

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Let $n = n_r = 2^r = m_r + 1 = m + 1$ and let the complete graph $K_n$ have its vertices denoted by the integers $\ell = 0, 1, \ldots, m_r$.

For $\ell = 1, \ldots, m_r$, let $\tau_\ell : V(Q_m) \to V(Q_m)$ be such that $\tau_\ell(x) = x + e_\ell^m$, for every $x \in V(Q_m)$, so that $(x, \tau_\ell(x)) \in f^\ell_m$.

**Proposition 2.1** The coclasses of the subgroup $H_m$ in the Abelian group $V(Q_m) = \mathbb{Z}^m_2$ different from $H_m$ coincide with the subsets $\tau_\ell(H_m)$ of $V(Q_m)$ obtained from $H_m$ by traversing the respective Cayley parallel 1-factors $f^\ell_m$, for $\ell = 1, \ldots, m_r$.

**Proof:** This uses the standard basis $\{e_1^m, \ldots, e_m^m\}$ of $V(Q_m) = \mathbb{Z}^m_2$ as a set of representatives of the nontrivial coclasses of $H_m$ in $\mathbb{Z}^m_2$. $\Box$

**Theorem 2.2** ([?]) There is a $2^{m-r}$-covering graph map $\pi_r : Q_m \to K_n$ such that $\pi_r^{-1}(0) = H_m$ and $\pi_r^{-1}(\ell) = \tau_\ell(H_m)$, for $\ell \in \{1, \ldots, m_r\}$.

We will think of $K_n$ as the target of the covering graph map $\pi_r$ in Theorem 2.2, (whose source is $Q_m$ meaning that $Q_m$ covers $K_n$) and consider $K_m \subset K_n$; more specifically, $K_m = K_n \setminus \{\emptyset\}$, that is: we take $V(K_m) = \{1, \ldots, m_r\}$.

**Corollary 2.3** The vertices of $K_m = K_n \setminus \{\emptyset\}$ are the targets of the cosets of $H_m$ different from $H_m$ through $\pi_r$.

The directing idea ahead is to refine the resulting partition of $\Sigma_m$ into the cosets of $H_m$ different from $H_m$ via weight parity as follows. Let the weight $wt(v)$ of a vertex $v$ of $Q_m$ be given via scalar product with the all-ones $m$-vector: $wt(v) = v \cdot (\Sigma_{\ell=1}^m e_\ell^m)$. Let $H^0_m$, (respectively, $H^1_m$), be the even-, (respectively, odd-), weight vertex set of $H_m$. The $2m_r$ subsets of $\Sigma_m$ obtained by splitting the subsets of the partition of $\Sigma_m$ into their intersections with $H^0_m$ and $H^1_m$ yield the cited refined partition. It is our aim to show that for $r > 2$ these $2m_r$ subsets constitute the vertices of two quotient graphs $\Pi_r, \Pi'_r$ of respective spanning subgraph $\Omega_r, \Omega'_r$ of $\Sigma_m$ which will be selfcomplementary in $\Sigma_m$. For the definition of the corresponding claimed covering graph maps, given via equations (1) below, we need the following algebraic notions.

We present the projective geometry $PG(r-1, 2) = GF_2^r \setminus \{0\}$ of dimension $r - 1$ over $GF_2$, (see for example [?], pg. 80), by identifying its $r$-tuples, say via the symbol ‘$\leftrightarrow$’, with powers of a root $x$ of a primitive irreducible polynomial $x^r + c_1x^{r-1} \ldots + c_r$ over $GF_2$, starting with the standard basis
given in reversed order with respect to usual setting, (see [?], pgs. 123-124): 00...001 ↔ \(x^0\), 00...010 ↔ \(x^1\), ..., 10...000 ↔ \(x^{r-1}\); then continuing with \(c_1...c_r \leftrightarrow x^r\); and so on for higher values of the integer exponent \(i\) of \(x^i \in PG(r - 1, 2)\), up to \(i = m_r - 1\), with the procedure going back to 00...001 ↔ \(x^{m_r} = x^0\), meaning that \(GF_2^r \setminus \{00...0\}\) is \(m_r\)-cyclic under multiplication.

The \(m_r\) \(r\)-tuples \(\lambda^r_i = t^r_1...t^r_i \in \mathbb{Z}_2^r\) with \(\lambda^r_i \leftrightarrow x^i\), where \(0 \leq i < m\), establish the 1-1 correspondence \(\lambda^r : \mathbb{Z}_m^r \rightarrow \mathbb{Z}_2^r \setminus \{\overline{0}\}\) that send \(i\) onto \(\lambda^r_i\). The \(m_r\)-sequence \(\Lambda_r = (\lambda^r_0, \lambda^r_1, ..., \lambda^r_{m_r-1})\) can be considered as a cycle graph by defining vertex adjacency just via contiguity in \(\Lambda_r\), where each term of \(\Lambda_r\) is considered contiguous to its predecessor and to its successor, with the additional convention that \(\lambda^r_{m_r-1}\) and \(\lambda^r_0\) are also contiguous.

Now we need the following elementary lemma, that furnishes the target \(K_n\) of \(\pi_r\) with the structure of an auxiliary undirected complete Cayley graph, and thus different from the Cayley structure used to define the source \(Q_m\).

**Lemma 2.4** \(K_n\) admits the structure of the undirected (complete) Cayley graph \(Kay^r_2 = Cay(\mathbb{Z}_2^r, \mathbb{Z}_2^r \setminus \{\overline{0}\})\) of the group \(\mathbb{Z}_2^r\) with respect to the generating set \(\mathbb{Z}_2^r \setminus \{\overline{0}\}\). In this way, the edges of \(K_n\) admit as labels the differences of their endvertex labels in \(\mathbb{Z}_2^r\).

**Proof:** The claimed admissible structure of \(Kay^r_2\) on \(K_n\) is obtained by renaming each vertex label \(\ell\) of \(K_n\) by means of its binary expression, considered as an element \(\lambda \in \mathbb{Z}_2^r\) via enough zeros to the left. \(\square\)

\(\Lambda_r\) can be seen as a Hamilton cycle of \(K_m\). In fact, Lemma 2.4 and the discussion preceding it yield \(\Lambda_r\) as an \(m_r\)-cycle in \(Kay^r_2\) with different edge labels and avoiding vertex \(\overline{0}\), (in a shifted order from the one in \(\Lambda_r\) as shown for \(r = 3\) in the Examples below).

**Notation.** Each of the \(m_r\) \(r\)-tuples \(\lambda^r_i \in \mathbb{Z}_2^r \setminus \{\overline{0}\}\) with \(\lambda^r_i \leftrightarrow x^i\), where \(0 \leq i < m\), could be read as an integer \(\ell^r_i\) from the leftmost nonzero entry of \(\lambda^r_i\) to the right. From now on, we use these shorthand nicknames \(\ell^r_i\) (that coincide with the original integer labels \(\ell\) of the vertices of \(K_m\)) in place of the corresponding \(\lambda^r_i \in \mathbb{Z}_2^r\). So, when representing \(K_n\) furnished with the structure of \(Kay^r_2\) as in Lemma 2.4, the labels of its edges and vertices \(\neq \overline{0}\) are to be cited as \(\ell^r_i\), meaning \(\lambda^r_i\). In the same way, the Steiner triple system \(L_r\) is to be presented by means of its integer nicknames instead than by the row vectors of \(H_r\). This shorthand notation was the main reason for our representing \(PG(r - 1, 2)\) with the standard basis taken in reversed
order, instead of the more usual direct top-to-bottom order of the rows of the identity \( r \times r \)-matrix.

**Examples.** We use the symbol ‘\( \equiv \)’ to relate each \( \ell^r_i \) with its corresponding \( \lambda^r_i \) and describe the presentation \( P_r \) of \( PG(r - 1, 2) \) given in this section by writing

\[
P_r = (\ell^r_i \equiv \lambda^r_i \leftrightarrow x^i; \ i = 0, 1, \ldots, m) = (1 \equiv 00 \ldots 01 \leftrightarrow x^0, \ 2 \equiv 00 \ldots 10 \leftrightarrow x^1, \ldots, \ \ell^r_{r-1} \equiv 10 \ldots 00 \leftrightarrow x^{r-1}, \ \ell^r_r \equiv c_1 c_2 \ldots c_r c_r \leftrightarrow x^r, \ldots, \ \ell^r_i \equiv \lambda^r_i \leftrightarrow x^i, \ldots),
\]

with integers expressed shortly in their hexadecimal notation, for \( r = 3, 4 \).

If \( P_3(x) = x^3 + x + 1 \), then \( P_3 = (1 \equiv 001 \leftrightarrow x^0, \ 2 \equiv 010 \leftrightarrow x^1, \ 4 \equiv 100 \leftrightarrow x^2, \ 3 \equiv 011 \leftrightarrow x^3, \ 6 \equiv 110 \leftrightarrow x^4, \ 7 \equiv 111 \leftrightarrow x^5, \ 5 \equiv 101 \leftrightarrow x^6) \). Thus,

\[
\Lambda_3 = (1, 2, 4, 3, 6, 7, 5).
\]

Recall that the Steiner triple system \( L_3 \) can be now expressed as:

\[
\{123, 145, 167, 246, 257, 347, 356\}.
\]

(So that the sequence of edge labels between 1 and 2, 2 and 4, etc. is \( (3, 6, 7, 5, 1, 2, 4) \), with a shifted order from that of \( \Lambda_3 \) and so is equivalent. See Figure 1(a).)

If \( P_4(x) = x^4 + x + 1 \), then \( P_4 = (1 \equiv 0001 \leftrightarrow x^0, \ 2 \equiv 0010 \leftrightarrow x^1, \ 4 \equiv 0100 \leftrightarrow x^2, \ 8 \equiv 1000 \leftrightarrow x^3, \ 3 \equiv 0011 \leftrightarrow x^4, \ 6 \equiv 0110 \leftrightarrow x^5, \ c \equiv 1100 \leftrightarrow x^6, \ b \equiv 1011 \leftrightarrow x^7, \ 5 \equiv 0101 \leftrightarrow x^8, \ a \equiv 1010 \leftrightarrow x^9, \ 7 \equiv 0111 \leftrightarrow x^{10}, \ e \equiv
\]
\(1110 \leftrightarrow x^{11}, \ f \equiv 1111 \leftrightarrow x^{12}, \ d \equiv 1101 \leftrightarrow x^{13}, \ 9 \equiv 1001 \leftrightarrow x^{14}\). Thus,

\[\Lambda_4 = (1, 2, 4, 8, 3, 6, c, b, 5, a, 7, e, f, d, 9).\]

Recall that the Steiner triple system \(L_4\) can be now expressed as:

\[
\left\{123, 145, 167, 189, 1ab, 1cd, 1ef, 246, 257, 28a, 29b, 2ce, 2df, 347, 356, 38b, 39a, 3cf, 3de, 48c, 49d, 4ae, 4bf, 58d, 59c, 5af, 5be, 68e, 69f, 6ac, 6bd, 78f, 79e, 7ad, 7bc\right\}.
\]

We enlarge \(\Lambda_r\) by interspersing, between each two contiguous terms \(\lambda', \lambda'' \in \Lambda_r\) with respective shorthand \(\ell'\), \(\ell''\), the element \(\lambda\) with shorthand \(\ell\) that makes up a triple \(\lambda'\lambda\lambda''\) (or \(\ell'\ell\ell''\)) in \(L_r\). Next, we modify the resulting sequence \((\gamma_r^0, \gamma_r^1, \ldots, \gamma_r^{2m-1})\), where \(\gamma_r^i = \lambda_r^i\) (or \(\ell_r^i\)), for \(i = 0, 1, \ldots, m - 1\), by setting alternate signs \(\pm\) to its terms, yielding the sequence \(\Gamma_r = (-\gamma_r^0, +\gamma_r^1, -\ldots, +\gamma_r^{2m-1})\). This sequence can be considered as a \(2^m r\)-cycle graph by defining vertex adjacency just via contiguity in \(\Gamma_r\), where each term of \(\Gamma_r\) is considered contiguous to its predecessor and to its successor, with the additional convention that \(+\gamma_r^{2m-1}\) and \(-\gamma_r^0\) are also contiguous. (See Figure 1(b) representing \(\Gamma_3\) corresponding to the \(\Lambda_3\) of Figure 1(a).)

Our objective in the next section is to extend \(\Gamma_r\) to a quotient graph \(\Pi_r\) of a spanning \(m_{r-1}\)-regular subgraph \(\Omega_r\) of \(\Sigma_m\) via a \(2^m r-r-1\)-covering graph map \(\theta_r : \Omega_r \rightarrow \Pi_r\) and defined on vertices to satisfy

\[
\theta_r^{-1}(-\ell) = \tau_{\ell}(\mathcal{H}_m^0) = \mathcal{H}_m^0 + e_\ell \quad \text{and} \quad \theta_r^{-1}(+\ell) = \tau_{\ell}(\mathcal{H}_m^1) = \mathcal{H}_m^1 + e_\ell, \quad (1)
\]

where \(\ell \in \{1, \ldots, m_r\}\), meaning that the inverse images of the \(2m_r\) vertices of \(\Gamma_r\) via \(\theta_r\) constitute a partition of \(V(\Sigma_m)\) formed by the even- and odd-weight-vertex halves of the coclasses of \(\mathcal{H}_m\) different from \(\mathcal{H}_m\). If \(\Pi'_r\) is defined from the claimed graph \(\Pi_r\) by exchanging the \(\pm\) signs while keeping the nickname labels \(\ell\), then for \(r > 2\) the edges of \(\Sigma_m\) not in \(\Omega_r\) will induce a subgraph \(\Omega'_r\) coincident with the inverse image of \(\Pi'_r\) via a \(2^m r-r-1\)-covering graph map \(\theta_r' : \Omega_r' \rightarrow \Pi_r'\) obtained by replacing \(\theta_r\) by \(\theta_r'\) in equations (1) above. Because of this, \(\Omega_r\) and \(\Omega'_r\) will become self-complementary regular subgraphs of \(\Sigma_m\).
3 Extending $\Gamma_r$

The desired extension $\Pi_r$ of $\Gamma_r$, needed for $r > 2$ since we can just take $\Pi_2 = \Gamma_2$, will be obtained by adjoining edges to $\Gamma_r$ so that the restrictions $\pi_r|\Omega_r$ and $\pi_r|\Omega'_r$ of $\pi_r$ to the desired subgraphs $\Omega_r$ and $\Omega'_r$ of $\Sigma_m$ will equal the compositions of $\theta_r$ and $\theta'_r$ with respective 2-covering graph maps $\psi_r : \Pi_r \to K_m = Kay_2^r \setminus \{\emptyset\}$ and $\psi'_r : \Pi'_r \to K_m = Kay_2^r \setminus \{\emptyset\}$ defined on vertices by $\psi_r(\delta \ell) = \psi'_r(\delta \ell) = \ell$, for $\delta = \pm$ and $\ell \in \{1, \ldots, m\}$, so we may say we are looking for sign-forgetful 2-covering graph maps $\psi_r, \psi'_r$ as required.

We proceed to adjoin edges to $\Gamma_r$ to fulfill our objective. This is done by means of the following steps and corresponding immediate implications. Of course, due to our remark on notation in Section 2, we mention integer labels that in reality are to be thought as their correspondent vectors in $\mathbb{Z}_2^r$.

We label each edge $e = (+u, -v)$ of $\Gamma_r$, (i.e. an edge between consecutive vertices in $\Gamma_r$), with the $w$ such that $uwv \in L_r$ and say that $uwv$ is the Steiner triple associated to $e$. Then, it is elementary that each vertex $+w$ of $\Gamma_r$ has neighbors $-u, -v$ in $\Gamma_r$ such that the labels of the edges $e_u = (+w, -u)$ and $e_v = (+w, -v)$ of $\Gamma_r$ are $v$ and $u$, respectively, where $uwv \in L_r$ is the Steiner triple associated both to $e_u$ and to $e_v$.

Observe that vertices $-1$ and $+1$ of $\Gamma_r$ are at distance $2r - 1$ in $\Gamma_r$.

We represent $\Gamma_r$ symmetrically on a circle $C$, as in Figures 2, 3, for $r = 4, 3$, respectively, where edge labels are shown in a smaller type to that of the vertex labels, in trying to avoid labelling confusion, and where vertices $\delta \ell$ are represented in black for $\delta = +$ and with a white interior if $\delta = -$, for better distinction. Consider the line $\ell_1$ that divides $C$ into two semicircles, as axis of the plane symmetry that exchanges the vertices $-1$ and $+1$. Now, the labels of the 4 edges incident to vertices $-i, +i$ in $\Gamma_r$ are pairwise different, for every $i \in \{1, \ldots, m_r\}$.

Let $D_r$ be a contiguity-preserving horizontal display of $\Lambda_r \setminus \{1\}$ with all its contiguous terms being equidistant. (See the Example below). Let $\chi_A$ be the reflection of the display $D_r$ with respect to its middle point $A$. Let $+a$ and $+b$, (respectively $-a'$ and $-b'$), be the neighbors of $-1$, (respectively $+1$), in $\Gamma_r$. Then, $a$ and $b$ are contiguous in $D_r$ and symmetrical to $a'$ and $b'$ with respect to $\chi_A$.

Let us associate a sign to each element $d \in D_r \setminus \{a, b, a', b'\}$ whose position is to the left of $A$, and let us associate the opposite sign to $\chi_A(d)$. Let this sign assignment be denoted by $\mathcal{A_r} : D_r \setminus \{a, b, a', b'\} \to \{\pm\}$. Then tracing a chordal edge in the circle representation of $\Gamma_r$ from $(-\mathcal{A_r}(d))1$ to $\mathcal{A_r}(d)d$,
for every $d \in \mathcal{D}_r \setminus \{a, b, a', b'\}$ provides vertices $-1, +1$ with incident edges having all labels in \{2, 3, \ldots, m_r\}. This is illustrated specifically in Figure 2 for the case treated in the Example below, where labels $\ell$ are written in hexadecimal integer notation.

Let us denote by $\Gamma^1_r$ the graph resulting from $\Gamma_r$ by adding the chordal edges referred to in the previous paragraph. Let us denote by $\Psi^1_r$ the set of edges of $\Gamma^1_r$ not in $\Gamma_r$. Then, $|\Psi^1_r| = n - 6$.

**Example.** Let $r = 4$. Consider the display $\mathcal{D}_r$ of $\Lambda_r \setminus \{1\}$ and the following sign assignment $\mathcal{A}_r$:

\[
\begin{align*}
\mathcal{D}_r &: \quad 2 \quad 4 \quad 8 \quad 3 \quad 6 \quad c \quad b \quad 5 \quad a \quad 7 \quad e \quad f \quad d \quad 9 \\
\mathcal{A}_r &: \quad - \quad - \quad - \quad - \quad - \quad + \quad + \quad + \quad + \quad + \\
\end{align*}
\]

which leads to the chordal edges represented in Figure 2. Thus, Figure 2 just shows $\Gamma_4 \cup \Psi^4_1$ for $r = 4$. 

Figure 3 shows two different edge selections for $\Psi^3_1$ (just those in double trace; the other chords exemplify in the following paragraph). This shows that the selection of $\Psi^3_1$ is not unique.
Let \( \Psi_r^r(\delta) \) be the subset of edges of \( \Psi_r^r \) incident to vertex \( \delta \), where \( \delta = \pm \).
Let \( \Psi_r^r(\delta) \) be the set of possible additional edges to \( \Gamma_r \) obtained from \( \Psi_r^r(\delta) \) by rotation from \( \delta \) to \( \delta \ell \), for \( \ell = 2, \ldots, m \), defined as in both cases considered in Figure 3, where we see that for \( r = 3 \), each \( \Psi_r^r(\delta) \) has cardinality 1.

Let \( \Psi_r^r(\delta) = \{ e_{1i}^r, \ldots, e_{is}^r \} \), where \( s = \frac{n-6}{2} \) and the order of these edges is given say with the remote endvertex from \( \delta \) running clockwise for \( \delta = + \) and counterclockwise for \( \delta = - \), out of \( \delta \). Then we can assume that \( \Psi_r^r(\delta) = \{ e_{1i}^r, \ldots, e_{is}^r \} \), where \( e_{1i}^r \) is obtained from \( e_{1}^r \) by means of the rotation that takes \( e_{1i}^r \) onto \( e_{1i}^r \), for \( i = 1, \ldots, s \). Let \( E_{i1}^{r\delta} = \{ e_{1i}^r : \ell = 1, \ldots, m \} \). Due to the preservation of the \( \Gamma_r \)-distances between the endvertices of the edges of \( E_{i1}^{r\delta} \), for \( \delta = \pm \), we conclude that \( E_{i1}^{r+} = E_{i1}^{r-} \), for each \( i = 1, \ldots, s \).

Since \( \bigcup_{i=1}^{m} \Psi_r^r(\delta) = \bigcup_{i=1}^{m} E_{i1}^{r\delta} \) for both \( \delta = \pm \), we conclude that the addition of all the edge subsets \( \Psi_r^r(+) \) to \( \Gamma_r \), for \( \ell = 1, \ldots, m \), yields the same supergraph \( \Pi_r \) of \( \Gamma_r \) as if we added instead all the edge subsets \( \Psi_r^r(-) \). (The reader may complete both these coincident rotational tasks in the case of Figure 2).

Let \( \Psi_r^r = \Psi_r^r(-1) \cup \Psi_r^r(+1) \). Then, the labels of edges incident to \( -\ell \) and \( +\ell \) in \( \Gamma_r \{ \{ -\ell, +\ell \} \} \) \( \Psi_r^r \subset \Gamma_r \) \( \Psi_r^r \) are pairwise different, and so they form the set \( \{ 1, \ldots, m_r \} \). This implies the existence of a \( 2^{m_r-1} \)-covering graph map \( \theta_r \) as claimed at the end of Section 2, stated as follows, so yielding our intermediate claim.

**Theorem 3.1** There exists a \( 2^{m_r-1} \)-covering graph map

\[
\theta_r : \Omega_r \to \Pi_r = \Gamma_r \cup \left( \bigcup_{i=1}^{m} \Psi_r^r(+) \right) = \Gamma_r \cup \left( \bigcup_{i=1}^{m} \Psi_r^r(-) \right),
\]

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where $\Omega_r$ is a spanning $(m - \frac{1}{2})$, (or $m_{r-1}$,), subgraph of $\Sigma_m$, so that the restriction $\pi_r|\Omega_r$ of $\pi_r$ to $\Omega_r$ equals the composition of a sign-forgetful 2-covering labelled-graph map $\psi_r : \Pi_r \to K_m$ with $\theta_r$, where

$$V(K_m) = \{1, \ldots, m_r\}$$

so that the label of each edge of $K_m$ together with those of its endvertices form a triple of $L_r$.

**Proof:** $\Omega_r$ is a spanning subgraph of $\Sigma_m$ with its edges defined as follows. An edge of $\Xi_r$ is of the form $(-i, +j)$. Then $\Omega_r$ contains all the edges of $\Sigma_m$ between $\tau_i(H^0_m)$ and $\tau_j(H^1_m)$. Repeat this process for every other edge of $\Xi_r$, which finally insures the well-definition of $\theta_r : \Omega_r \to \Pi_r$. Also notice that the labelled-graph map $\psi_r : \Pi_r \to K_m$ is well defined, since each vertex $\delta_i$ of $\Xi_r$ has an incident edge labeled with each integer $j \in \{1, \ldots, m_r\} \setminus \{i\}$ and so it does project effectively onto a corresponding edge of $K_m$ labelled with $j$ by means of $L_r$. This insures that $\psi_r : \Pi_r \to K_m$ is a 2-covering graph map that composes with $\theta_r : \Omega_r \to \Pi_r$ to yield $\pi_r : \Omega_r \to K_m$, as claimed. $lacksquare$

**Corollary 3.2** The graph $\Pi'_r$ obtained from $\Pi_r$ by exchanging the labels in each pair $\{-i, +i\}$ of vertices while keeping edge labels in $\Pi'_r$ as in $\Pi_r$ is isomorphic to the quotient graph of a sign-forgetful $2^{m_r-r-1}$-covering labelled-graph map $\theta'_r$ whose domain is the complementary graph $\Omega'_m$ of $\Omega_r$ in $\Sigma_m$ if $r > 2$ and coincides with $\Omega_r$ if $r = 2$. Moreover, if $r > 2$ then $\Omega_r$ and $\Omega'_r$ constitute an equitable $m_{r-1}$-factorization of $\Sigma_m$.

**Proof:** Assume just that $r > 2$. The edges of $\Sigma_m$ not in $\Omega_r$ induce the graph $\Omega'_r$. On the other hand, Theorem 3.3 assures that the edges of $\Omega_r$ represent only half of the edge classes between even- and odd-weight-vertex halves of coclasses of $\mathcal{H}_r$ different from $\mathcal{H}_r$. Then, the $2^{m_r-r-1}$-covering graph map $\theta'_r : \Omega'_r \to \Pi'_r$ is well defined and provides the necessary complementary assignment of edge classes to those remaining edges of $\Sigma_m$ not in $\Omega_r$, that is the remaining classes of edges between those coclass halves. $lacksquare$

Notice for $r > 2$ that $\theta_r^{-1}(\Gamma_r)$ is the disjoint union of $2^{m_r-r-2}$ 2$m_r$-cycles and that the restriction of $\theta_r$ to each one of these cycles is a graph isomorphism. Each edge subset $\Psi'_r(\delta)$ works as a set of ‘ladders’ between the different ‘levels’ provided by the resulting $2^{m_r-r-2}$ components of $\theta_r^{-1}(\Gamma_r)$.
4 Equitable 1-Factorizations of $\Sigma_{2r-1}$, for $r \geq 2$

The corollary below is true elementarily for $r = 2$, so we proceed to establish it for $r > 2$. We say that a 1-factor $F$ of $\Pi_r$, $(\Pi'_r)$ is an assortment if there is a one-to-one correspondence $\xi : F \to \{1, \ldots, m_r\}$ such that $\xi(e)$ is the edge label obtained by $e \in F$ in the construction procedure of Section 3.

**Proposition 4.1** There exists a 1-factorization $F$, $(F')$, of $\Pi_r$, $(\Pi'_r)$, in which each 1-factor is an assortment.

**Proof:** The claimed $F$ can be attained for example by taking: a complete set of alternative edges in the outer cycle (on the circle $C$) of the adopted representation of $\Gamma_r$, $\Pi_r$ and $\Pi'_r$ to form one of the component 1-factors, say $F_1 \in F$; the remaining edges of this outer cycle $\Gamma_r$ as another component 1-cycle, say $F_2 \in F$; and for each selection of one edge in each $\Psi_r(\delta)$, with $\delta$ fixed and say equal to $-$, for $\ell$ running from $\ell = 1$ to $\ell = m$, so that the resulting set $\Phi$ of $m$ edges is invariant under the group of rotations of $C$, this $\Phi$ as $F_i \in F$, for $i = 3, \ldots, m$. □

It is clear that the pullback $E' = (\theta_r)^{-1}(F)$, $(E'' = (\theta'_r)^{-1}(F'))$, is a 1-factorization of $\Omega_r$, $(\Omega'_r)$, with its component 1-factors intersecting each $f_m^\ell$ at exactly a constant number of edges, namely $2^{m_r-r-1}$. Clearly, the union of both 1-factorizations, $E = E' \cup E''$, is a 1-factorization of $\Sigma_m = \Omega_r \cup \Omega'_r$, with the same equitable property, which yields our desired result, with the same number $2^{m_r-r-1}$ of edges in the intersection of each 1-factor of $E$ with each $f_m^\ell$ but twice the number of component 1-factors with respect to $E'$ or $E''$.

**Corollary 4.2** For $r \geq 2$ and $m_r = 2^r - 1$, there exists a 1-factorization of $\Sigma_m$ having the property that its 1-factors intersect each Cayley parallel 1-factor of $Q_m$ at exactly a constant number of edges, namely $2^{m_r-r-1}$ edges.

Recall that each Cayley parallel 1-factor $f_m^\ell$ separates $Q_n$ into two $m_r$-subcubes $Q^{0}_{m}$ and $Q^{1}_{m}$, where $Q^{i}_{m}$ is induced by $\{v \in V : v_i = k\}$, for $k = 0, 1$. Let $\rho_n : Q_n \to Q_m$ be given by $\rho(x_1, \ldots, x_m, x_n) = (x_1, \ldots, x_m)$. Then, if $E$ is a 1-factor of $E$ and if $E' = E \cap \Omega_r$, $E'' = E \cap \Omega'_r$, it can be checked easily that $(\rho_n^{-1}(E') \cap Q^{0}_{m}) \cup (\rho_n^{-1}(E'') \cap Q^{1}_{m}) \cup Q_n[\rho^{-1}(H_m)]$, is a 1-factor of $Q_n$ with the same number of edges along each coordinate direction of $Q_n$ and such that each one of its maximal subset of parallel edges has minimum distance 3, a result obtained in [?] and used in the establishment of its main result.
5 Conclusions

Combinatorial and geometric properties of the $m$-cube $Q_m$ are still at large and quite unknown. The Hamming shell $\Sigma_m$ ($m = m_r = 2^r - 1$) is of interest as well for areas of computer applications like interconnection networks and parallel computing. In fact, $\Sigma_m$ constitutes a codegree one subgraph of $Q_m$ thus being an $(m - 1)$-regular subgraph sharing in addition quite a lot of its inherent symmetry. In this context, it becomes natural to pursue any knowledge about objects like say our equitable factorizations. This knowledge bank would be at the disposition of researchers looking for better ways of communication and methods of calculation involved in complex problems.

The previous paragraph contains the practical motivation behind our explicit construction above of a 1-factorization of $\Sigma_m$ in $Q_m$, for $r \geq 2$, having the discussed equitable property, with $2^{m_r} - r - 1$ edges along each Cayley parallel 1-factor of $Q_m$ and of an equitable $m_{r-1}$-factorization of $\Sigma_m$ formed by two factors which are spanning regular subgraphs, self-complementary in $\Sigma_m$ generalizing what was already known for $r = 3$ with respect to the Foster graph, even though it may be contended that these objects are inherently interesting by themselves.

The tools in our construction involved linear Hamming codes, Steiner triple systems and the subjacent finite algebra mod 2 typical of such combinatorial designs and coding theory applications. However, the outcome of this research step pertains more to the related combinatorial area of graph theory. In fact, the treated concept of an equitable factorization refers to the equitative distribution of factors between distinct factorizations of cubes seen as Cayley graphs, and so it is related to equitable labelings of graphs, a generalization of graceful labelings, one of the research areas dear to Professor Alexander Rosa. On the other hand, the adequate labeling of $Q_m$ and $\Sigma_m$ for this relation is done by means of the standard $m$-tuple notation mod 2 of their vertices instead that by means of an interval of the positive integers, as is the typical case for graceful and equitable labelings of graphs.

For the continuation of this line of research, possible subsequent problems include studying the structure of cycles in bipartite subgraphs of $\Sigma_m$ formed as the union of any two factors of the obtained 1-factorization, as well as the determination of the maximum degree of a subcube contained in $\Sigma_m$ and of the maximal number of such pairwise disjoint subcubes in $\Sigma_m$. Any properties that may be substracted from the core of these useful Cayley
graphs and subgraphs are of potential interest for technological developments.

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**References**


