Efficient Dominating Sets in Cayley Graphs

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Abstract

An independent set C of vertices in a graph is an efficient dominating set (or perfect code) when each vertex not in C is adjacent to exactly one vertex in C. An E-chain is a countable family of nested graphs, each of which has an efficient dominating set. The Hamming codes in the n-cubes provide a classical example of E-chains. We give a constructing tool to produce E-chains of Cayley graphs. This tool is used to construct infinite families of E-chains of Cayley graphs on symmetric groups. These families include the well-known star graphs, for which the efficient domination property was proved by Arumugam and Kala, and pancake graphs. Additional structural properties of the E-chains and the efficient dominating sets involved are also presented. Given a tree T, the T-graph associated to T seems to be a natural candidate of a graph with an efficient dominating set. However, we prove that a T-graph has an efficient dominating set if and only if T is a star.

1 Introduction and Notation

Let $\Gamma = (V, E)$ be a finite undirected graph with no loops and multiple edges. We follow the terminology of [5]. Given $C \subseteq V$, let the *open neighborhood* $N(C)$ of C in Γ be the subset of vertices in $V \setminus C$ adjacent to some vertex in C, and let the corresponding *closed neighborhood* be $N[C] = N(C) \cup C$. A set $C \subset V$ is a *dominating set* if $N[C] = V$, that

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is, every vertex in Γ is adjacent to some vertex in C. The domination number $\gamma(\Gamma)$ is the minimum cardinality of a dominating set in Γ. If the dominating set C is a stable set of Γ, then C is an *independent dominating set.* Also, when every vertex in $V \setminus C$ is adjacent to exactly one vertex in C then C is a *perfect dominating set.* A dominating set C which is both independent and perfect is an *efficient* dominating set. In what follows we may refer to an efficient dominating set as an E-set for short.

E-sets correspond to perfect 1-correcting codes in Γ, as treated by N. Biggs [4] and J. Kratochvil [7]. Equivalently, they provide a perfect packing of Γ by balls of radius 1. When Γ is r-regular, the so-called *sphere packing* condition

$$|V| = (r+1)|C|$$

is trivially a necessary condition for C to be an E-set of Γ.

When the set of vertices is $V = F^n$, where F is an alphabet of q symbols, and two n-tuples in V are adjacent whenever they agree in all but one coordinate, E-sets are the standard 1-perfect codes. For any group structure given to F, the resulting graph can be viewed as a Cayley graph, which we denote by $F(q,n)$. When q is a prime power, the classical Hamming codes show that the sphere packing condition, expressible now as

$$q^n = 0 \pmod{n(q-1)+1},$$

is also sufficient for the existence of E-sets in $F(q,n)$.

We say that a countable family of graphs

$$\mathcal{G} = \{\Gamma_1 \subset \Gamma_2 \subset \ldots \subset \Gamma_i \subset \Gamma_{i+1} \subset \ldots\}$$

is an E-*chain* if every Γ_i is an induced subgraph of Γ_{i+1} and each Γ_i has an E-set C_i.

For graphs Γ and Γ', a one-to-one graph homomorphism $\zeta : \Gamma \to \Gamma'$ is an *inclusive map* if $\zeta(\Gamma)$ is an induced subgraph of Γ'.

Let κ_i stand for the inclusive map of Γ_i into Γ_{i+1} induced by $\mathcal{G}$, for $i \geq 1$. If $C_{i+1} = N(\kappa_i(V_i))$, where V_i is the vertex set of Γ_i, then we say that the E-chain $\mathcal{G}$ is a *neighborly* E-chain.

If, for each $i \geq 1$, there exists an inclusive map $\zeta_i : \Gamma_i \to \Gamma_{i+1}$ such that $\zeta_i(C_i) \subset C_{i+1}$, then we say that the E-chain $\mathcal{G}$ is *inclusive.* Notice that an inclusive neighborly E-chain has $\kappa_i \neq \zeta_i$, for every positive integer i.

A particular case of inclusive Γ_i is the one in which C_{i+1} has a partition into images $\zeta_i^{(j)}(C_i)$ of C_i through respective inclusive maps $\zeta_i^{(j)}$, where j varies on a suitable finite indexing set. In such a case, the E-chain $\mathcal{G}$ is said to be *segmental.*

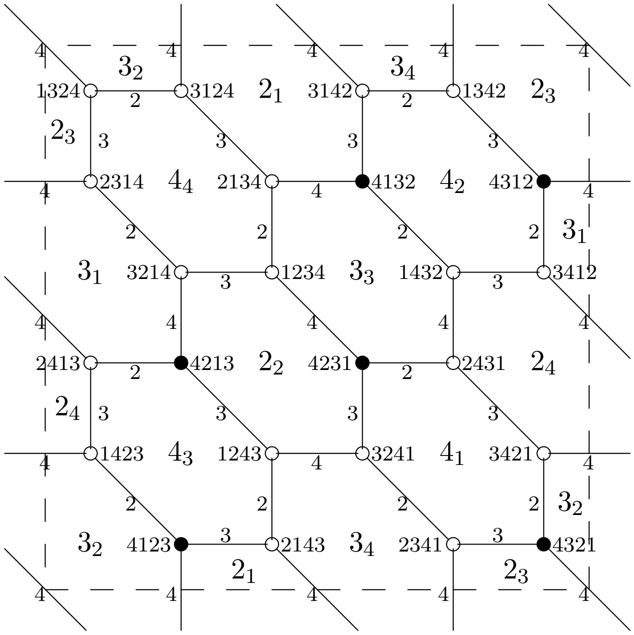

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Figure 1: The star graph ST_4 can be embedded in the torus.

The family of Hamming cubes $\{F(2, 2^n - 1),\ n \geq 1\}$ provides an example of a segmental E-chain. An example of a segmental neighborly E-chain is given by the star graphs (or star-transposition graph) $ST_n,\ n \geq 1$. The star graph ST_n is the Cayley graph on the symmetric group on n letters with respect to the set of transpositions $\Sigma = \{(1\ i),\ i = 2, \ldots, n\}$. It was shown by Arumugam and Kala in [2] that the graph ST_n do possess an E-set. Moreover, they showed that the E-set of ST_n contains an E-set of ST_{n-1} for each $n \geq 2$.

Figure 1 shows the graph ST_4 with the vertices of one of its E-sets indicated in black, where vertex notation is shown without parentheses and commas and where edges and copies of ST_3 (these given as hexagonal faces of the shown toral embedding) are labeled as follows:

(a) if $v = v_1v_2v_3v_4$ and $w = w_1w_2w_3w_4$ are the endvertices of an edge e in ST_4, then there is a unique integer entry i with $1 < i \leq 4$ such that $v_i \neq w_i$, which we use as a label for e;

(b) for each $i = 2, 3, 4$ and $j = 1, 2, 3, 4$, a copy of $ST(3)$ in $ST(4)$, that we denote i_j, is given by the set of all vertices $v = v_1v_2v_3v_4$ of $ST(4)$ with $v_i = j$; the 4 6-cycles labeled i_j avoid the edges labeled i, which form themselves a 1-factor.

Items (a) and (b) are extensible to any ST_{n+1}. In particular, the given definition of copies of $ST(3)$ in $ST(4)$ denoted i_j, is immediately extensible to the definition of copies i_j of ST_n in ST_{n+1}, for $i = 2, \ldots, n+1$ and $j = 1, \ldots, n+1$. Then, observe that $N(i_j)$ is an E-set of ST_{n+1}, in every feasible instance, for $|V(ST_{n+1})| = (n+1)!$ and $|N(i_j)| = n!$. Thus, $\gamma(ST_{n+1}) = n!$.

Notice that the E-set shown in Figure 1 can be made segmental in 3 different ways, via 3 respective inclusive maps from ST_3 into ST_4, because for $k = 1, 2, 3$ and $j = 2, 3, 4$, there are embeddings $\zeta_3^{k,j} : ST_3 \to ST_4$ with $\zeta_3^{k,j}(ST_3) = j_k$, such that the black vertices in j_k form the image of an E-set in ST_3 through $\zeta_3^{k,j}$; this way, the 3 maps $\zeta_3^{k,j}$, with j fixed, play the role of the maps $\zeta_i^j = \zeta_3^j$ in the definition of a segmental E-chain. This extends immediately for any ST_{n+1}.

Note that if $\mathcal{G}$ is an E-chain of regular connected graphs (and Γ_1 is 1-regular as we may assume without loss of generality), then the sphere packing condition implies $|V(\Gamma_n)| \geq (n+1)!$, $n \geq 1$. Therefore, the E-chain of the star graphs $\Gamma_n = ST_{n+1}$ is as 'dense' as such a chain can be, because $|V(ST_{n+1})| = (n+1)!$, $(n \geq 2)$, whereas the E-chain of the Hamming cubes is rather 'sparse'. Moreover, any corresponding E-set C_n of $\Gamma_n = ST_{n+1}$ satisfies $|C_n| = n!$. An E-chain $\mathcal{G}$ is said to be *dense* if, for each $n \geq 1$, one has $|V(G_n)| = (n+1)!$ and $|C_n| = n!$.

In Section 2, we give a general constructing tool for dense segmental neighborly E-chains, or DSNE-chains. We apply this tool to produce an uncountable collection of countable families of Cayley graphs, among them the star graphs and the pancake graphs, of such DSNE-chains in Section 3. We also show that the vertex sets of the resulting graphs can be partitioned into E-sets, a feature shared with the classical Hamming codes. The star graphs belong to a general class of so-called tree graphs, see Section 4 for the definition. We show in that section that, given a tree T, the T-graph has an E-set if and only if T is a star.

2 Constructing Tool

Recall that the Cayley graph of a group G with respect to a subset $A \subset G$, which we denote by $Cay(G, A)$, has the elements of the group as vertices and there is an edge $\{x, y\}$ whenever $x^{-1}y \in A \cup A^{-1}$. We will always assume that A does not contain the identity element $1 \in G$, so that the resulting graph has no loops. We also assume that $A = A^{-1}$.

Cayley graphs are regular of degree $|A|$; they are connected if and only if A generates G and they are vertex-transitive. In particular, the left translations $\phi_a(x) = ax$ are automorphisms of the graph. Notice that, for every subset X of vertices in the Cayley graph $Cay(G, A)$, it holds that $N(X) = XA \setminus X$ and $N[X] = XA'$, where $A' = A \cup \{1\}$.

The construction of E-chains of Cayley graphs is based on the following lemma.

Lemma 1 *Let A be a generating set of a finite group G such that $s^2 = 1$ for each $s \in A$. Let $u \in A$ be such that $A_u = A \setminus \{u\}$ generates a proper subgroup H of G of index $(|A|+1)$ in G.*

If $U = H \cap uHu$ is an E-set in $Cay(H, A_u)$, then the open neighborhood $N(H)$ of H is an E-set in $Cay(G, A)$.

Moreover, there are inclusive maps $\zeta^{(j)}$ such that $\{\zeta^{(j)}(U),\ j = 1, \ldots, |A|\}$ is a partition of the E-set $N(H)$.

Proof. Notice that U is the intersection of H and its conjugate by u. Therefore, U is a subgroup of G contained in H.

For each $x \in G$, let Γ_x be the subgraph of $\Gamma = Cay(G, A)$ induced by the vertices in xH. In particular, $\Gamma_1 = Cay(H, A_u)$ and, for each $x \in G$, $\Gamma_x = \phi_x(\Gamma_1)$, where ϕ_x denotes the left translation by x, which is a graph isomorphism.

By hypothesis U is an E-set of Γ_1, which is an $(|A|-1)$-regular graph. The sphere packing condition gives $|H| = |A| \cdot |U|$.

Let $D = N(H)$ be the open neighborhood of H in Γ. We shall prove that D is an E-set of Γ. Since $u \notin H$ and $H = \langle A_u \rangle$, we have $D = Hu$.

For each $x \in G$, let $D_x = D \cap xH$. If $D_x \neq \emptyset$, then we may assume that $x = hu$ for some $h \in H$ and we have

$$D_x = Hu \cap xH = (hu)(uHu \cap H)xU = xU = \phi_x(U). \tag{1}$$

In particular, the number of left cosets intersected by D is $|D|/|D_x| = |H|/|U| = |A|$. Since $|G| = (|A|+1)|H|$, all cosets of H except H itself are intersected by D. Moreover,

$$\begin{aligned} N(D) &= DA \setminus D = (DA_u \cup Du) \setminus D = (\cup_{x \in G \setminus H}(D_x A_u) \cup H) \setminus D \\ &= (\cup_{x \in G \setminus H}(xH) \cup H) \setminus D = G \setminus D, \end{aligned}$$

where we have used the fact that $D_x = \phi_x(U)$, and therefore D_x is an E-set of Γ_x for each $x \in G$. Since D has the right cardinality, $|G|/(|A|+1)$, it is an E-set of Γ.

To prove the last statement of the Lemma, note that $Uu = (H \cap uHu)u = Hu \cap uH = uU$. Let $T = \{x_1, \dots, x_{|A|}\}$ be a right transversal of U in H. Then,

$$D = Hu = (\cup_{x_j \in T} x_i U)u = \cup_{x_j \in T} x_j uU.$$

Therefore, by defining $\zeta^{(j)}$ as the restriction to Γ_1 of the graph isomorphism $\phi_{x_j u}$, we get the desired set of inclusive maps. $\square$

3 E-chains of Cayley Graphs

The lemma above provides a tool to produce DSNE-chains of Cayley graphs. We use it below to obtain such chains of Cayley graphs on the symmetric groups.

Let σ_i be the transposition $(1\ i)$ and let π_i be an arbitrary product of transpositions on the set $\{2, 3, \dots, i-1\}$, $(1 < i)$, where $\pi_2 = \pi_3$ are defined to be the identity permutation, which we denote by ι. For each positive integer $n \geq 2$ let

$$A(\pi_2, \dots, \pi_n) = \{\sigma_2\pi_2, \dots, \sigma_n\pi_n\}.$$

Lemma 2 *For each positive integer $n \geq 2$ and any choice of the involutions π_i, $i \geq 4$, the set $A(\pi_2, \dots, \pi_n)$ generates the full symmetric group S_n.*

Proof. Assume that $A(\pi_2, \dots, \pi_{n-1})$ generates the full symmetric group S_{n-1}, for some $n > 2$. Since $\sigma_n\pi_n(n) = 1$, then the group G_n generated by $A(\pi_2, \dots, \pi_n)$ acts transitively on $\{1, \dots, n-1, n\}$. The stabilizer of n in G_n has cardinality $(n-1)!$ and therefore $|G_n| = n!$. Thus, $G_n = S_n$. The result follows by induction on n. $\square$

For each choice of the involutions $\pi_2, \pi_3, \dots$, with $\pi_i \in Sym(2, \dots, i-1)$, the sequence of Cayley graphs Γ_n on the symmetric group S_{n+1} with respect to the generating set $A(\pi_2, \dots, \pi_{n+1})$ forms a chain of nested graphs with the natural inclusions $\Gamma_n \subset \Gamma_{n+1}$. This class of graphs include some well-known families of Cayley graphs. For instance, if we choose the identity permutation for each of the π_i's, then Γ_n is the star graph ST_{n+1}. When $\pi_i = (2\ (i-1)) \cdots (\lfloor i/2 \rfloor\ \lceil i/2 \rceil)$, $i = 4, \dots, n+1$, we obtain the *pancake graph* PC_{n+1}, see for instance [1]. The first three members of the family, PC_1, PC_2 and PC_3, coincide with the star graphs ST_1, ST_2, ST_3, respectively. The pancake graph PC_4 is represented in Figure 2, where edges are labeled as in Figure 1, but here we are able to represent copies of PC_3 only in 4 instances, which , with a notation similar to that of Figure 1, can

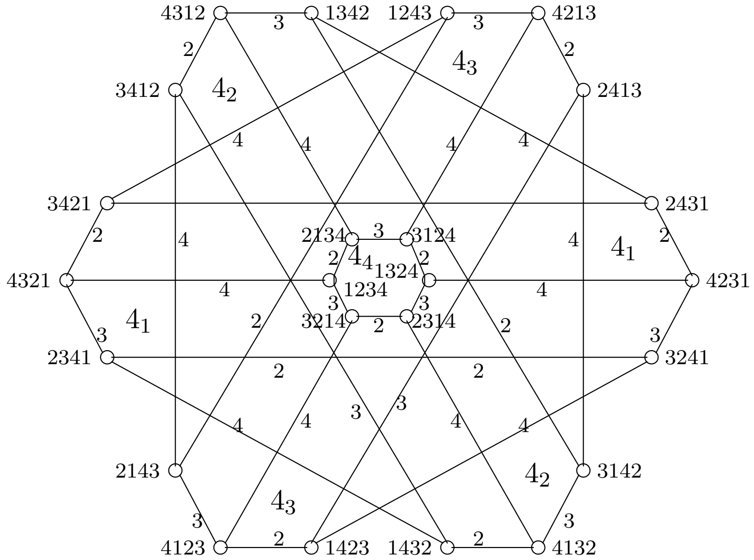

Figure 2: Representation of the pancake graph PC_4.

be denoted $4_1, 4_2, 4_3, 4_4$, given by the 4 6-cycles whose vertices have their last entry equal to the subindex i of the respective 4_i, for $i = 1, 2, 3, 4$.

We next show that the families obtained in this way form DSNE-chains.

Theorem 3 *Let* $\Pi = \{\pi_2, \pi_3, \pi_4, \cdots\}$ *be a family of involutions with* $\pi_i \in Sym(2, \ldots, i-1)$ *for each* $i \geq 4$ *(and* $\pi_2 = \pi_3 = \iota$*). Let* Γ_n *be the Cayley graph on the symmetric group* S_{n+1} *with respect to the set of permutations* $A^{(n+1)} = A(\pi_2, \ldots, \pi_{n+1})$*,* $n \geq 1$*.*

Then the family $\mathcal{G}^{\Pi} = \{\Gamma_n,\ n \geq 1\}$ *is a DSNE-chain.*

Proof. Let H_{n+1} be the stabilizer of $n+1$ in the symmetric group S_{n+1} on

$\{1, 2, \ldots, n+1\}$. We shall prove, by induction on n, that $N(H_{n+1})$ is an E-set of Γ_n.

The statement clearly holds for $n = 1$. For $n > 1$, we apply Lemma 1 with $u = \sigma_{n+1}\pi_{n+1}$ and $H = H_{n+1}$, which is the subgroup generated by $A_u^{(n+1)} = A^{(n+1)} \setminus \{u\} = A^{(n)}$. Clearly $H_{n+1} = S_n$ and has index $n+1 = |A^{(n+1)}| + 1$ in S_{n+1}. Note that uHu is the set of permutations in S_{n+1} which fix 1. Thus, $U = H \cap uHu$ is the set of permutations of S_{n+1} which fix 1 and $n+1$ pointwise. Therefore, if $v = \sigma_n\pi_n$, then $U = vH_nv$.

We can write $vH_nv = \phi'_v(N'(H_n))$, where ϕ'_v denotes the left translation by v in the Cayley graph $\Gamma_{n-1} = Cay(S_n, A^{(n)})$, and $N'(H_n)$ denotes the open neighborhood of H_n in this graph Γ_{n-1}. By the induction hypothesis, $N'(H_n)$ is an E-set of G_{n-1}. Since ϕ'_v is a graph isomorphism, then $U = \phi'_v(N'(H_n))$ is also an E-set in this graph. By Lemma 1, $N(H)$ is an E-set of G_n.

This shows that $\{\Gamma_n,\ n \geq 2\}$ is a neighborly, inclusive and dense E-chain. By the second statement of Lemma 1, the E-chain is also segmental. $\Box$

Example. Consider the star graph ST_4. In this case we have $u = (1\ 4)$ and

$$H = \{(1\ 2\ 3\ 4), (2\ 1\ 3\ 4), (3\ 1\ 2\ 4), (1\ 3\ 2\ 4), (2\ 3\ 1\ 4), (3\ 2\ 1\ 4)\},$$

giving its elements in the order appearing in the 6-cycle that constitutes H_{S_u}. (In particular, observe that H is the vertex set of 4_4 in Figure 1.) Then by listing the elements of uHu in the same order as the corresponding elements of H, we have that

$$uHu = \{(1\ 4\ 2\ 3), (1\ 2\ 4\ 3), (1\ 3\ 4\ 2), (1\ 4\ 3\ 2), (1\ 2\ 3\ 4), (1\ 3\ 2\ 4)\},$$

so that the first and fourth vertices in the representation of H appear as the fifth and sixth vertices in the representation of uHu. This is due to the fact that the fifth and sixth elements of H have its third component $a_3 = a_{n-1}$ equal to 1, which produces that $b_4 = 4$ in the corresponding elements of uHu through ξ. This way, as a result of Lemma 1, $N(4_4)$, indicated in Figure 1 as the black vertices, is an E-set. $\Box$

Theorem 3 above provides a wide range of examples of DSNE-chains on the symmetric groups. Indeed, since there is an uncountable collection of choices for the family Π, there is an uncountable collection of examples of countable families of DSNE-chains including the star graphs (when $\pi = \iota$ for each i) and the pancake graphs (when $\pi_i = (2\ (i-1)) \cdots (\lfloor i/2 \rfloor\ \lceil i/2 \rceil),\ i \geq 4$).

The E-chains described above share an additional interesting property with the classical Hamming codes concerning partitions of the vertex set into E-sets.

Given a graph $\Gamma = (V, E)$, if $C^{(1)}, \ldots, C^{(r)}$ are pairwise disjoint E-sets of G such that $V = \cup_{i=1}^{r} C^{(i)}$, then we say that $\{C^{(1)}, \ldots, C^{(r)}\}$ is an E-partition of Γ. An E-chain $\mathcal{G}$ is said to be *split* if each of its component graphs, say Γ_i, has a partition into E-sets $C_i^{(1)}, C_i^{(2)}, \ldots, C_i^{(r_i)}$. If $r_i = i$, (resp. $r_i = 2^i$), for every positive integer i, then we say that the E-chain $\mathcal{G}$ is *linearly*, (resp. 2-*exponentially*), split. The classical Hamming codes constitute a 2-exponentially split E-chain. However, nonlinear Hamming codes may yield nonsplit E-chains.

Theorem 4 *For every collection* $\Pi = \{\pi_2, \pi_3, \pi_4, \cdots\}$ *of involutions with*

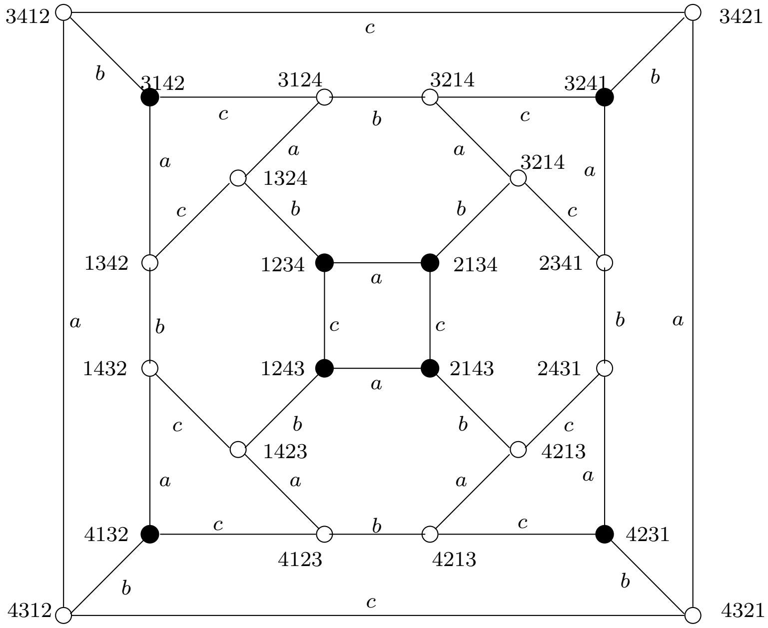

Figure 3: The 3-path graph Γ_{P_3} is a truncated octahedron graph.

$\pi_i \in Sym(2, \ldots, i-1)$ *for each* $i \geq 4$ *(and* $\pi_2 = \pi_3 = \iota$*), the E-chain* $\mathcal{G}^\Pi$ *is linearly split.*

Proof. Fix $n \geq 1$. As it is shown in the proof of Theorem 3, if $u = \sigma_{n+1}\pi_{n+1}$ and $H = H_{n+1}$ denotes the stabilizer of $n+1$ in S_{n+1}, then $C_n = N(H) = Hu$ is an E-set of Γ_n.

For each pair of elements $x, y \in S_{n+1}$ we have $|xHu \cap yHu| = |xH \cap yH|$. Therefore, $xC_n \cap yC_n$ is either empty or $xC_n = yC_n$. Since the left translations ϕ_x are graph isomorphisms of Γ_n, its vertex set can be partitioned into the E-sets xC_n for $x \in S_{n+1}$. Since $|C_n| = n!$, the chain $\mathcal{G}^\Pi$ is linearly split. □

4 Tree-Transposition Graphs

The star graphs belong to the general family of tree graphs, (or tree-transposition graphs), whose hamiltonicity was proved in [6, 8]. Given a tree T on n vertices labeled by the integers $1, 2, \ldots, n$, for each edge $e = ij$ in T let τ_e be the transposition $(i\ j)$ in the symmetric group S_n of permutations on n symbols. It is well-known, ([3]), that the resulting set of transpositions, which we denote by $\mathcal{T} = \{\tau_e,\ e \in E(T)\}$, generates the entire symmetric group S_n. The T-graph, that is the tree graph associated to T, is the Cayley graph $\Gamma_T = \Gamma_{\mathcal{T}}$ on S_n with respect to the transpositions in $\mathcal{T}$. When T is a star with $n-1$ leaves, the resulting graph is the star graph ST_n. On the other hand, Figure 3 shows the smallest tree graph different from a star graph. For any sequence $T_1 \subset T_2 \subset \ldots \subset T_n$ of trees such that T_i is obtained from T_{i+1} by deleting one of its leaves, the corresponding tree graphs $\Gamma_{T_1} \subset \Gamma_{T_2} \subset \ldots \subset \Gamma_{T_n}$ form a chain of graphs, each of which satisfies the sphere-packing condition. However, they form an E-chain if and only if T_n is a star.

Theorem 5 *Given a tree T, the T-graph has an E-set if and only if T is a star.*

Proof. Let T be a tree with n vertices different from a star. Then T has an internal edge, say jk (i.e. j and k are not leaves of T), and $T - jk$ is the disjoint union of two trees T_1 and T_2 with n_1 and n_2 vertices respectively.

By the sphere packing condition, an E-set of Γ_T would have cardinality $(n-1)!$. Let us see that this is not possible.

Let $\Gamma_{T'}$ be the Cayley graph on S_n with respect to the set of transpositions in $\mathcal{T}' = \mathcal{T} \setminus \{(j\ k)\}$, that is, Γ_T is the edge-disjoint union of $\Gamma_{T'}$ and the 1-factor of all edges associated to the transposition $(j\ k)$.

Lemma 6 *The subgraph $\Gamma_{T'}$ of Γ_T is (isomorphic to) the disjoint union of $\binom{n}{n_1}$ copies of the graph product $\Gamma_{T_1} \times \Gamma_{T_2}$.*

Proof. Label the vertices of T_1 with $1, \ldots, n_1$ and the vertices of T_2 with $n_1+1, \ldots, n_1+n_2$. Let Γ_1 be the connected component of $\Gamma_{T'}$ containing the identity permutation $12\ldots n$. Then the map $\phi : S_{n_1} \times S_{n_2} \to S_n$ which sends $(x_1 \ldots x_{n_1}, y_1 \ldots y_{n_2})$ to $x_1 \ldots x_{n_1} y'_1 \ldots y'_{n_2}$, where $y'_i = n_1 + y_i,\ i = 1, \ldots, n_2$, is easily checked to be a graph isomorphism from $\Gamma_{T_1} \times \Gamma_{T_2}$ to Γ_1. Since $\Gamma_{T'}$ is a vertex symmetric graph of order $n!$, then it consists of $\binom{n}{n_1}$ vertex disjoint copies of $\Gamma_{T_1} \times \Gamma_{T_2}$. □

Example. For $n = 5$, consider T with vertex set $\{1, 2, 3, 4, 5\}$ and edges 13,23,34 and 45. Then the component of $\Gamma_T[E_T \setminus 34]$ containing vertex 12345 has vertex sets:

$$\{12345, 32145, 31245, 21345, 23145, 13245,$$
$$12354, 32154, 31254, 21354, 23154, 13254\},$$

whose first six shown vertices form a 6-cycle, or copy of ST_3, and so do the last 6, and where the i-th and $(i+6)$-th vertices form a K_2, or copy of ST_2, for $1 \leq i \leq 6$. Since in this case Γ_{T_1} and Γ_{T_2} are respectively ST_3 and ST_2, we see that the cited component is their graph product. By permuting the identity vertex 12345 in the vertex set above respectively by 12435, 12534, 13425, 13524, 14523, 23415, 23514, 24513 and 34512, which are all the different possibilities of selecting the three first numbers from 1 to 5, or equivalently, of selecting the last two numbers, we get the vertex sets of all the $\binom{5}{2} = 10$ components of $\Gamma_T[E \setminus 34]$, all isomorphic to $ST_3 \times ST_2$. □

Returning to the proof of Theorem 5, suppose that S is an E-set of Γ_T. Then, S intersects the connected component Γ_1 of the spanning subgraph $\Gamma_{T'}$ in a set S_1 of vertices which are at distance at least 3 one from each other. By Lemma 6, this connected component is isomorphic to the graph product $\Gamma_{T_1} \times \Gamma_{T_2}$. Therefore, the projections of S_1 onto each one of the factors Γ_{T_1} and Γ_{T_2} have at most cardinalities $(n_1-1)!$ and $(n_2-1)!$ respectively. Therefore $|S_1| \leq (n_1-1)!(n_2-1)!$ and

$$|S| \leq \binom{n}{n_1}(n_1-1)!(n_2-1)! = \frac{n!}{n_1 n_2}.$$

Since $\min\{n_1, n_2\} \geq 2$, we have $|S| < (n-1)!$, contradicting that S is an E-set. □

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