A note on Frucht diagrams, Boolean graphs and Hamilton cycles

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Abstract


We use techniques based on F-diagrams to determine Hamilton cycles in some Boolean graphs.

Let $i,j \in \mathbb{Z}$, with $0 < i < j$, and let $n = i + j$. Let $RG_{ij}$ be the graph induced by the middle levels of the Boolean algebra of the set $\{0, 1, \ldots, n-1\}$. A generalization of the conjecture of Havel [6] (also, Kelly [7], and attributed to Erdős in [2]) is that every $RG_{ij}$ is Hamiltonian. Here we will show that some quotients have the following structure: Let $\Gamma$ be a finite cyclic group. Let $\Gamma$ act on a finite graph $G$. Frucht [5] defined an F-diagram of $G$ as a quotient graph of $G$ under the action of $\Gamma$. In dealing with that conjecture, we found that F-diagrams can be extended [4], in the family of graphs in the conjecture, so as to set its claim for $i = j-1 < 9$, $i = j-2 < 7$ and $i = j-3 = 6$ [2-4]. The aim here is to sketch our application of F-diagrams for the Hamilton cycle search in $RG_{ij}$, with a concrete example for $i = j-1 = 5$.

Let $\Gamma$ be a group and let $G = (V,E)$ be a graph. A $\Gamma$-action $\tau = (T,t)$ on $G$ is a pair of actions $T : \Gamma \times V \rightarrow V$ and $t : \Gamma \times E \rightarrow E$ such that $t(k, e)$ has ends $T(k, u)$ and $T(k, v)$, for every $ke\Gamma$ and $eeE$ with ends $u, v \in V$. Let $V/T$ be the quotient set of $V$ under $T$. Let $\Gamma$ be a finite cyclic group and let $\tau = (T,t)$ be a $\Gamma$-action on a finite graph $G$. The quotient graph $G/\tau$ is a graph $(V/T, E/t)$ with a graph map $G \rightarrow G/\tau$, i.e. subjacent maps $V \rightarrow V/T$, $E \rightarrow E/t$ and satisfying (a) and (b) below. Select $b$ in each $B \in V/T$. Let $b' \in B$.

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Then $b' = T(z, 1, b)$ for some $z \in Z$. We set $b' = B, z$. We orient arbitrarily each $e_i \in E/t$ into an arc $\hat{e}_i$.

(a) For each $e_i \in E/t$, if $\hat{e}_i$ joins $B \in V/T$ to $C \in V/T$, then there is an $e \in E$ of $e_i$, with $B, 0$ and $C, z$ as its ends, for some integer $z$ in $[0, d)$, where $d = \gcd \{|B|, |C|\}$. We label $\hat{e}_i$ with $z$.

(b) The graph $(V/t, E/t)$ does not depend on the orientation selected for $E/t$. If the orientation of an arc $\hat{e}_i$ with label $z$ is reversed, then the label of the reversed arc is $-z + d$.

The graph $G/\tau$ with an orientation of $E/t$, an arc labelling and a labelling in $V/T$ by the $T$-orbit cardinalities will be called an $F$-diagram. Such an $F$-diagram is denoted by $G^f$.

A $\Gamma$-action $\tau = (T, t)$ on a graph $G$ is said to be free if both $T$ and $t$ are free. We denote the edge labels above as $F$-voltages. Let $\Gamma$ be a finite solvable group, i.e. $\Gamma$ has a cyclic tower of subgroups $\{e\} = \Gamma_m \subseteq \Gamma_{m - 1} \subseteq \cdots \Gamma_2 \subseteq \Gamma_1 \subseteq \Gamma_0 = \Gamma$ such that each $\Gamma_{k + 1}$ is normal in $\Gamma_k$ and the group $\Sigma(k) = \Gamma_k/\Gamma_{k + 1}$ is cyclic, for every $k \in I_m = \{0, 1, \ldots, m - 1\}$. We can generalize an $F$-diagram for any $\Gamma$-action $\tau: \Gamma \times G \to G$ on a graph $G$: Consider the induced $\Sigma(m - 1)$- action $\tau(m - 1)$ of $\tau$ on $G$. The corresponding quotient graph $G/\tau(m - 1)$ admits both the structure of an $F$-diagram $G^{m - 1} = G^{\Sigma(m - 1)}$ and an induced $\Sigma(m - 2)$-action $\tau(m - 2)$. A series of $F$-diagrams $G^{m - 1}, G^{m - 2}, \ldots, G^0$ can be defined this way. $\Gamma$ is obtainable by semidirect products from the cyclic subgroups $\Sigma(m - 1), \Sigma(m - 2), \ldots, \Sigma(0)$. We get a copy of $\Sigma(k)$ in $\Gamma$ and a $\Sigma(k)$-action $\tau_k = (T_k, t_k)$ on $G$ as a restriction from $\tau$.

If $\{b, c\} \subseteq I_m$, then $\tau_b$ induces an action

$$\tau^b_b = (T^b_k, t^b_k): \Sigma(b) \times (G/\tau_c) \to G/\tau_c,$$

given by $T^b_k(k, T_b(\Sigma(c), d)) = T_b(\Sigma(c), T_b(k, d))$, where $k \in \Sigma(b)$ and $d \in G/\tau_c$. Its quotient graph is denoted by $G/\tau_c/\tau_b$. Similarly, if $\{a, b, c\} \subseteq I_m$, then $\tau_a$ induces an action

$$\tau^a_a = (T^a_a): \Sigma(a) \times G/\tau_c/\tau_b \to G/\tau_c/\tau_b.$$

In this case, three quotient graphs can be taken successively from $G$ under the actions $\tau_a, \tau_b$ and $\tau_c$. No matter in which order we take the three successive quotient graphs from $G$, this process will always yield the same quotient graph $G/\tau_c/\tau_b/\tau_a$. Moreover, let $\alpha_b: G \to G/\tau_b, \alpha_c: G/\tau_b \to G/\tau_c/\tau_b$ and $\alpha^{ab} a: G/\tau_c/\tau_b \to G/\tau_c/\tau_b/\tau_a$, etc., be the quotient graph maps corresponding, respectively, to the actions $\tau_b, \tau^a_b$ and $\tau^{ab}_b$. Each square diagram made out of these quotient graph maps commutes. Any application induced by some $\tau_a$ as above, like $\tau^b_a$, is also denoted by $\tau_b$.

We can take $RG_{ij}$ as a bipartite graph having vertex classes $D_i$ and $D_j$ formed by the $n$-tuples whose Hamming weights are $i$ and $j$, respectively. Let $r$ be a fixed positive integer with $\gcd(r, n) = 1$. Certain actions on vector coordinates yield group actions $\tau_s = (T_s, t_s)$ on $RG_i (s = 0, 1, 2, 3)$ that we determine as follows:

(0) Let $T_0$ be complementation, i.e. a $Z_2$-action determined by

$$T_0(1, d) = (1 + d_0, \ldots, 1 + d_{n - 1}) \quad (d \in D_i \cup D_j).$$
(1) Let $T_1$ be reversal, i.e. a $\mathbb{Z}_2$-action determined by

$$T_1(1,d) = (d_{n-1}, \ldots, d_0) \quad (d \in D_i \cup D_j).$$

(2) Let $T_2$ be rotation or shifting, i.e. a $\mathbb{Z}_n$-action given by

$$T_2(k, d) = (d_{-k}, d_{-k+1}, \ldots, d_{-1}) \quad (k \in \mathbb{Z}_n, \ d \in D_i \cup D_j).$$

(3) Let $T_3$ be $r$-interleaving, i.e. a $\mathbb{Z}_\mu$-action ($\mu$=least positive $m \in \mathbb{Z}$ such that $r^m \equiv 1 \mod n$) determined by

$$T_3(-1, d) = (d_0, r, d_{2r}, \ldots, d_{-r}) \quad (d \in D_i \cup D_j).$$

Another graph we have here, denoted by $G_{ij}$, is defined as having the same vertex subset that $RG_{ij}$ has, and its adjacency differing from the one of $RG_{ij}$ in that two vertices $u$, $v$ of $RG_{ij}$ are adjacent in $RG_{ij}$ iff $u$ and $T_1(v)$ are adjacent in $G_{ij}$. The actions on $G_{ij}$ corresponding to those above will be denoted, by an abuse of notation, with the same symbols $\tau_i = (T_i, i)$ ($i = 0, 1, 2, 3$).

We remark that all the Hamilton cycles obtainable by our methods in the $RG_{ij}$ can be translated into Hamilton cycles in the corresponding $G_{ij}$ by only changing the vertices of $D_j$ via $T_2(1, -)$-reversing.

Let $R\tau_0 = (R\tau_0, R\tau_0)$ be reversed complementation, i.e. the $\mathbb{Z}_2$-action obtained as the composition $\tau_0 \tau_1$ of the actions $\tau_0$ and $\tau_1$. We indicate between parentheses the respective alternatives. The subgroup of the automorphism group of $RG_{ij}$ ($G_{ij}$) generated by the groups of the actions $R\tau_0$, $\tau_2$ and $\tau_3$ ($\tau_0$, $\tau_2$ and $\tau_3$) will be denoted by $R\Gamma$ ($\Gamma$).

We will describe the generalized F-diagrams obtainable from the $R\Gamma$- and $\Gamma$-actions on $RG_{ij}$ and $G_{ij}$, respectively, and how these F-diagrams have a common underlying graph which is a quotient for both $RG_{ij}$ and $G_{ij}$. According to Havel [6], Laborde observed that the quotient graph $G_{ij}/\tau_0$ is the Kneser graph $O_{i,j}$, defined in $[1]$ as having its vertex set equal to the family of subsets of $I_n$ whose cardinality is $i$ and for which two vertices are adjacent if they have an empty intersection in $I_n$.

The graph $RO_{ij} = RG_{ij}/R\tau_0$ has the same vertex set that $O_{ij}$ has, but its adjacency differs from that of $O_{ij}$ in the same way as the adjacencies of $RG_{ij}$ and $G_{ij}$ differed, just by label reversal, that is, by the action of $T_1 = R$.

Every graph $G = (V, E)$ has associated with it a bipartite $G' = (V', E')$, with vertex set $V' = V \times \{0, 1\}$ formed by classes $V \times \{0\}$ and $V \times \{1\}$. It is easy to check that $RG_{ij}$ ($G_{ij}$) is isomorphic to $RO_{ij}$ ($O_{ij}$), or, by means of a joint notation that we will use subsequently, $(R)G_{ij}$ is isomorphic to $(R)O_{ij}$.

We remark that $RG_{ij}/R\tau_0$ ($G_{ij}/\tau_0$) is isomorphic to $RO_{ij}$ ($O_{ij}$) and identify $RG_{ij}/R\tau_0$ with $RO_{ij}$, and also $G_{ij}/\tau_0$ with $O_{ij}$. Since $(R)\Gamma$ is obtainable by semidirect products from cyclic sub-groups, thus being solvable, the earlier conclusions on successive quotients by means of $\tau_0, \ldots, \tau_m$ hold for the present actions $(R)\tau_0$, $\tau_2$ and $\tau_3$. We agree that notations $\Gamma, G, O, \tau_0$, etc., respectively, stand either for
RF, RG_{ij}, RO_{ij}, RT_0, etc., or for G, GI_{ij}, O_{ij}, \tau_0, etc. Let

\[ H = O/\tau_2 \text{ and } J = H/\tau_3. \]

We may recover, from the notations G, O', O, H, H', J and J', the original notations, prefixing R if appropriate and indexing with \( i \).

Let \( A \) be a vertex of \((R)H_i\). Then \( A \) is a \( T_2 \)-orbit:

\[ A = (a_0 \ a_1 \cdots \ a_{n-1} \ a_1 \ a_2 \cdots \ a_{n-1} \cdots \ a_{n-1} a_0 \cdots a_{n-2}) \]

in \( D_j \cup D_{j+1} \). The dot notation used in the definition of an F-diagram will be replaced, for the action \( \tau_2 \), by colon notation. We will select, for each \( A \in (R)H_i \), a distinguished representative \( A:0 \in D_j \cup D_{j+1} \) in the corresponding \( T_2 \)-orbit.

Suppose that \( A:0 = a_0 a_1 \cdots a_{n-1} \). We use the notation \( A:1 = a_1 \cdots a_{n-1} a_0, \ldots, A:n-1 = a_{n-1} a_0 \cdots a_{n-2} \), so that \( A = (A:0 \ A:1 \cdots \ A:n-1) \). We also use the notation \( A = (A:0) \) to show the particular selection of \( A:0 \).

A vertex of \((R)H_j\) is said to be palindromic iff some representative of it in \( D_j \) is \( T_0 \)-invariant. A vertex \( A = (A:0) \) of \( RH_{ij} \) has a symmetry pivot at its coordinate \( a_0 \) if each pair of coordinates \( a_s \) and \( a_{n-s} \) have the same value for \( s = 1, \ldots, [n/2] \). Let \( a^s \) be the vertex of \( RH_{ij} \) having a representative \( (a_0 a_1 \cdots a_{n-1}) \), with \( a_k = 1 \) iff \( 0 \leq k \leq j \). We remark that a vertex of \( RH_{ij} \) is palindromic iff it has a symmetry pivot. In particular, \( a^s \) is palindromic.

**Theorem 1** (Dejter [2]). If \( RG_{ij} \) has \( T_2 \) free, then, in order to get a Hamilton cycle \( Q \) in \( RG_{ij} \) by cyclotomic lifting from a Hamilton path \( S \) in \( RH_{ij} \), it is sufficient that \( S \) has as ends \( a^s \) and another vertex with an incident loop, and, in the case that \( n \) is composite, that \( S \) has two interior vertices adjacent by means of a 2-link.

The vertices \( (a_0, \ldots, a_{n-1}) \) of \( RH_{ij} \) or \( H_{ij} \) will be called necklaces and represented by lowercase letters. While \( RT_0, T_0 \) and \( T_2 \) are free, so that their orbits are of the same cardinality, \( T^{32}_3 \) may yield different orbit cardinalities.

For \( n \) equal to a prime power, \( \tau_0(R\tau_0) \) is a restriction of \( \tau_3 \) for \( RG_{ij}(GI_{ij}) \). Thus, it is irrelevant to set the prefix R in front of the symbol J in equations (*), so that \( RJ_{ij} = J_{ij} \). But furnishing this graph with F-voltages from \( RH_{ij} \) or from \( H_{ij} \) differs. Let \( oc \) stand for \( T^{32}_3 \)-orbit cardinality. Capital letters represent vertices of \((R)J_{ij}\). If \( A \) is such a vertex and if \( A.0 = (a) \) then we use the notation \( A = [a] \).

Let \( PRJ_{ij} \) be the subset of palindromic vertices of \( RJ_{ij} \). There is a maximal \( oc \) for the vertices of \( RJ_{ij} \) (\( PRJ_{ij} \)), that we denote by \( moc(n) \) (\( pmoc(n) \)). Either \( pmoc(n) = 1/2 moc(n) \) or \( pmoc(n) = moc(n) \) according to whether \( \tau_0 \) is a restriction of \( \tau_3 \) or not. Let \( j = i + 1 \) from now onwards. We try to find a subgraph of \( RJ_{ij} \) consisting of a collection \( \{P\} \) of disjoint paths with the ends (interior vertices) of each \( P \) having \( oc = pmoc(n)(moc(n)) \). If \( \tau_0 \) is a restriction of \( \tau_3 \), as in the example \( i = 5 \) below, then (i) each \( P \) as in (a) above can be blown up to \( pmoc(n) \) directed cycles \( P.\mu \) in \( RH_{ij} \), and (ii) we may get a path by splitting these cycles \( P.\mu \) into paths and plugging to their ends the remaining vertices. These cycles are taken with orientations given in their
Frucht diagrams, Boolean graphs and Hamilton cycles

descriptions, from left to right, so that if we write \( P_\mu = (\ldots, P_s, \mu', P_{s+1}, \mu', \ldots) \), where \( \mu' = \mu \) or \( \mu + \text{mo}(n) \), then we adopt the notation \( P_{s+1} \mu' \) \((s \cdot \mu' = (P_{s+1} \cdot \mu')^{-1})\) for \( P, \mu = (P_s, \mu', P_{s+1}, \mu') \) as an oriented path from \( P_{s+1}, \mu' \) to \( P_s, \mu' \). (P.s to \( P_{s+1}, \mu' \)), where lowercase letters, like \( p \), replace capital letters, like \( P \).

Example i = 5. For \( RJ_{5,6} \) with \( r = 2 \), (a) is applicable with the only path \( P = (P_1, \ldots, P_5) \), with \( P_1 = \rightarrow [10111010000], \ P_2 = \leftarrow [01000111111], \ P_3 = \rightarrow [10111100001], \ P_4 = \leftarrow [01000111110], \ P_5 = \rightarrow [11111000001] \). \( P \) blows up to five cycles \( P.s = (P_1.s, P_2.s, \ldots, P_5.s, P_4.s(s+5), \ldots, P_2.(s+5), P_1.s) \) for \( s = 0, \ldots, 4 \in Z_{10} \). Then the representatives of the remaining vertex \( A = [11011100010] \) in \( RJ_{5,6} \) (oc = 2) can be integrated into a path \( (\pm p.0, A.0, \pm p.2, \pm p.1, A.1, p.5.9) \).

Note added in proof. Orbital representations of graphs were introduced by R. Frucht in 1970. Later on, in 1977, H. Hévia generalized these representations proving that each cyclic subgroup of automorphisms of a graph \( G \) determines one of such representations for \( G \). (See: Representacion orbital de grafos y numero minimo de puntos para grafos n-ciclicos, Scientia 148 (1977) 102–122.) W. Arlinghaus in 1985 was the first to introduce the name of F-diagram to an orbital representation of graphs. (See: ‘The classification of minimal graphs with given abelian automorphism group’, Memoirs of the Amer. Math. Soc. 57 no. 330 (1985).)

References