A number of examples of perfect dominating sets of n-cubes are produced whose induced components are subcubes of the same dimension but occurring parallel to different hyperplanes, by means of the structure of n-cubes in relation to Steiner triples and Hamming codes. One of these examples, given in the 8-cube, has as components lines surprisingly in all coordinate directions. This suggest the conjecture that the total number of different hyperplanes to which the r-cube components of a perfect dominating set are parallel is either 1 or 4 or 8, and that, with the exclusion of the mentioned example in Q8 and its extensions to higher dimensional cubes, a lower bound for the number of coordinate directions in which these components occur is \((n/2)+r-1\).

1. Introduction.

Given a graph \(\Gamma\) and given a set of vertices \(S\) of \(\Gamma\), we say that \(S\) is a perfect dominating set (PDS) of \(\Gamma\) if every vertex of \(\Gamma\) is either in \(S\) or is adjacent to exactly one vertex of \(S\).

The n-dimensional cube \(Q_n\) is taken as the graph whose set of vertices is \((\mathbb{F}_2)^n\) or \(\{0,1\}^n\) and such that two vertices are adjacent if they differ in exactly one coordinate. Vectors representing vertices of \(Q_n\) will be denoted either \((h_1,h_2,\ldots,h_n)\) or \(hh\ldots nh\), without parentheses or commas if no confusion arises. Another notation for an n-cube vertex to be used is Rado's, that expresses in a sequence only the coordinate indices of the corresponding vector that have weight or value 1, where the empty symbol \(\emptyset\) is used to represent the null vector.

We continue the work of the second author in [7], in which he proved that a PDS of the n-cube \(Q_n\) induces a subgraph of \(Q_n\) whose components are isomorphic to hypercubes, and if \(Q_r\) is such a

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component then \( n-r \equiv 1 \text{ or } 3 \mod 6 \). The subgraph induced by a PDS will be called also a PDS, meaning now perfect dominating subgraph. If a PDS is a set of independent edges, we say that it is a linear PDS (LPDS).

The \( n \)-cube \( Q_n \) has a set of canonical 1-factorizations whose 1-factors are composed each by a set of parallel edges. In fact, consider all the edges of \( Q_n \) whose endvertices differ only in the first coordinate. We say that this edge set is the first parallel 1-factor \( f_n \) of \( Q_n \). In the same way, we define the \( i \)-th 1-factor \( f_i \) of \( Q_n \), for \( i=2, \ldots, n \).

Given two parallel 1-factors \( f_i \) and \( f_j \) of \( Q_n \), with \( i<j \) say, they give rise to a parallel 2-factor \( f_i \cup f_j \) formed by 4-cycles, along exactly those parallel 1-factors. Similarly, parallel \( r \)-factors \( f_{i_1} \cup \cdots \cup f_{i_r} \) formed by \( r \)-cubes along the \( r \) parallel 1-factors \( f_{i_1}, f_{i_2}, \ldots, f_{i_r} \) can be defined.

The examples of PDS's in [7] are perfect Hamming codes in the \( Q_n \)'s for which \( n \) is 1 less than a power of 2, and those obtained from these by taking products with (repeated) copies of \( K_2 \). Clearly, the resulting PDS's are the products of perfect Hamming codes with a product of \( K_2 \)'s, that is with a smaller-dimensional cube. But the components of these PDS's are all parallel to a common hyperplane of \( Q_n \). In this paper, we consider the case of PDS's whose components are subcubes of the same dimension but not parallel to a common hyperplane. This type of PDS will be referred as a twisted PDS's.

In searching to overcome the parallelism constraint in the cited examples of [7], Felzenbaum [5] produced a twisted LPDS in \( Q_8 \) whose components are 32 edges of \( Q_8 \) belonging, in groups of 8, to four different parallel 1-factors of \( Q_8 \) (See Section 2 below).

Using a method to construct twisted LPDS's in the \( 2^r \)-cube with components along \( 2^r \) parallel 1-factors inspired in Felzenbaum's construction (See Section 3) and some ideas arising from [2,3,4] we were able to produce:

1. a twisted LPDS in \( Q_{16} \) whose components are \( 2^{14} \) edges of \( Q_{16} \) belonging, in groups of \( 2^8 \), to eight different parallel 1-factors of \( Q_{16} \) (See Section 5).

Independently, the ideas of [2,3,4] yield in Section 4:

2. a twisted LPDS in \( Q_8 \) whose components are 16 edges of \( Q_8 \) belonging, in groups of two, to the eight parallel 1-factors of \( Q_8 \).

However, this types of constructions seem to fail for higher powers of 2 other than \( 2^8 \) and \( 2^{16} \), because higher powers-of-two
cubes do not "clutch" or "match" as well as \( Q_8 \) and \( Q_{16} \) due to the intricacy of their geometries. Thus, we conjecture the following.

**Conjecture 1.** Given a PDS \( \theta \) of an \( n \)-cube whose components are \( r \)-cubes, the number \( m \) of parallel \( r \)-factors along which the components of \( \theta \) happen can only be \( m = 1 \) or \( 4 \) or \( 8 \).

In order to support this conjecture, using the two constructions above, we were able also to produce twisted PDS's which are not just products of smaller dimensional PDS's by copies of \( K_2 \):

(a) A twisted PDS in \( Q_{4i+1} \) whose components are \((i+1)\)-cubes in four different parallel \((i+1)\)-factors, for \( i = 1,2 \);

(b) A twisted PDS in \( Q_{5i+1} \) whose components are \((i+1)\)-cubes in eight different parallel \((i+1)\)-factors, for \( i = 1,2,3,4,5,6 \).

However the method that produced them, to be found in Section 6, cannot be extended for values of \( i \) higher than those given above.

A second conjecture refers to the number of coordinate directions that a PDS in an \( n \)-cube uses as is based on the evidence given by the constructions of this paper.

**Conjecture 2.** Given a PDS \( \theta \) of \( Q_n \) whose components are \( r \)-cubes, and such that \( \theta \) is not as in (1) above or obtained from it by products by copies of \( K_2 \), an upper bound for the number of coordinate directions of \( Q_n \) in which the components of \( \theta \) occur is \((n/2)+r-1\).

2. Felzenbaum's LPDS with components in Four Parallel 1-Factors of \( Q_6 \)

A pair of opposite edges in \( Q_n \) is a pair of edges such that the endvertices of one of them is transformed into the endvertices of the other one by complementation, that is by adding the \( n \)-vector \( 11...1 \) of weight \( n \) to both endvertices.

There is a 1-factor \( F_4 \) in \( Q_4 \) formed by two opposite edges in each one of the four parallel 1-factors of \( Q_4 \). A more convenient description of \( F_4 \) is as follows: \( Q_4 \) can be viewed as the edge-disjoint union of two copies of \( Q_3 \) and a parallel 1-factor. This parallel 1-factor can be taken to be \( F_4 \) (notation as in Section 1) so that the two said copies of \( Q_3 \) are the subcubes \( Q'_3 \) induced by those vertices of \( Q_4 \) represented by vectors whose last coordinate is \( i \) for \( i=0,1 \).

Consider the projection \( p \) of \( Q_4 \) onto \( Q_3 \), sending a 4-vector into the 3-vector formed by its first three coordinates. Consider the
Hamming code $\Phi_0$ in $Q_4$ formed by 000 and 111. The complement $Q_4-\Phi_0$ intersects $Q_4\cup Q_4$ on a pair of 6-cycles. We take a 1-factor $f$ of this pair of cycles. Moreover, we can take $f$ as formed by pairs of opposite edges in each parallel 1-factor of $Q_4$. The convenience of doing so is that $f$ can then be completed to a 1-factor of $Q_4$ by the addition of a pair of opposite edges in $f_4$. This is the 1-factor $F_4$ we were looking for and it is formed by four opposite edge sets $E_1$, $E_2$, $E_3$, and $E_4$, say respectively along the coordinate directions 1, 2, 3, and 4.

Let us represent the vertices of the 8-cube by 8-tuples $uvwxijkm$, where $u, v, w, x, i, j, k, m \in \{0, 1\}$. The 8-cube $Q_8$ can be taken as the product $Q_4Q_4$, and as such, as codified by the first $Q_4$ in the product, to be denoted from now on as a "big 4-cube", each one of whose "big vertices" $uvwx$ represents the copy of the second $Q_4$ in the product with first four coordinates fixed on the respective values $u, v, w$ and $x$. Such a big vertex is denoted as the "small 4-cube" $Q_4(uvwx)$. Each one of the "big edges" of the big $Q_4$ between corresponding adjacent big vertices represents then a complete set of parallel edges between the represented small 4-cubes.

A big vertex $Q_4(uvwx)$ can be taken as a 4-cube with 16 vertices of the form $uvwxijkm$, with $i, j, k, m$ varying. There are four big edges departing from this big vertex (in four different coordinate directions!), the first one of which takes those 16 vertices to the 16 ones of the form $u'vwxijkm$, where $u'$ is the complement of $u$ and so on. Consider the set $Odd_4$ of all vertices $uvwx$ with odd weight $u + v + w + x$ in the big $Q_4$. Subdivide $Odd_4$ into four sets $V_1$, $V_2$, $V_3$, $V_4$ of two opposite vertices each. Then Felzenbaum's LPDS is formed by taking a copy of $E_i$ inside the small 4-cubes represented by each one of the two big vertices of $V_i$ for $i=1, 2, 3, 4$. Thus, two edges along each of the four parallel 1-factors are used in this twisted LPDS.

Figure 1 depicts the selection of the $E_i$'s and the $V_i$'s.


A 1-factor of $Q_n$ is said to be irreducible if it does not have parallel edges in any 2-subcubes (or 4-cycle or subsquare) of $Q_n$, and if it intersects all parallel 1-factors of $Q_n$. Notice that the 1-factor in the big $Q_4$ in Section 2 is irreducible in this sense.

In the attempt of generalizing Felzenbaum's example, we thought of constructing an irreducible 1-factor of $Q_n$, where $n=2^r$, as follows.

Consider $Q_n$ as $Q_{n-1} \cup Q_{n-1} \cup f_n$, the edge-disjoint union of
the graphs $Q^n_{n-1}$ induced by all the vertices of $Q_n$ having the last coordinate equal to $i = 0,1$ respectively and the $n$th parallel 1-factor $f^1_n$ of $Q_n$.

For $i=1,2,...,n-1$, let $\tau_i$ be the automorphism of $Q_{n-1}$ that complements the $i$th coordinate of the vertices of $Q_{n-1}$.

It is known that $Q_{n-1}$ contains a perfect (Hamming) code $C_0$ formed by the solutions of an equation $H \cdot x = 0$, where $H$ is an $r_x(n-1)$-matrix whose columns are the binary expressions of the integers $1,2,...,n-1$ written from top to bottom instead that from left to right; $x$ is an unknown $(n-1)a_1$-matrix and $C_0$ is the null $r_x1$-matrix. The vertices of such a perfect code will be distinguished as Hamming vertices.
The translation of such a perfect code \( \Phi \) in \( Q_n \) along the \( i \)-th parallel 1-factor, which is the image of \( u_i \), will be also considered as a perfect code \( \Phi_i \), for \( i = 1, \ldots, n-1 \).

Let \( m_{Q_n} \rightarrow Q_{n-1} \) be a projection that eliminates the \( i \)-th coordinate of \( n \)-tuples representing vertices of \( Q_n \), for \( i = 1, \ldots, n \), with \( m_{u}(u_1, u_2, \ldots, u_n) = u_1 u_2 \cdots u_{n-1} \), for each vertex \( u_1 u_2 \cdots u_n \) of \( Q_n \).

The subgraph \( E_0 \) of \( Q_n \) induced by \( (m_{n})^{-1}(\Phi) \) is a matching contained in the \( n \)-th parallel 1-factor of \( Q_n \). We want to complete \( E_0 \) to an irreducible 1-factor of \( Q_n \). In order to do this, we must look for a collection of pairwise vertex-disjoint matchings \( E_0, E_1, \ldots, E_{n-1} \) (\( E_0 \) already given above) such that \( E_i \) is the subgraph induced by \( (m_{n})^{-1}(\Phi_i) \), for \( i = 1, \ldots, n-1 \).

If we have at our disposal a collection of matchings \( E_i \) as just described, then to emulate Felzenbaum's argument, we consider \( Q_{2n} \) as \( Q_n \times Q_n \), a big \( Q_n \) whose vertices represent each a small \( Q_n \). Then, by finding adequate vertex sets \( V_0, V_1, \ldots, V_{n-1} \) in the big \( Q_n \), a twisted LPDS might be obtained by taking a copy of \( E_i \) in the small \( n \)-cube represented by each vertex of \( V_i \) for \( i = 0, 1, \ldots, n-1 \).

It is known ([2,3,4]) that there exists a covering graph map \( \Phi: Q_{n-1} \rightarrow K_n \) given as follows. Assume that the vertices of \( K_n \) are denoted \( 0, 1, 2, \ldots, n-1 \). Then we take \( \Phi^{-1}(i) \) equal to the Hamming code \( \Phi_i \) in \( Q_{n-1} \), for each \( i = 0, 1, \ldots, n-1 \).

We split \( \Phi \) into \( \Phi_{\text{even}} \) and \( \Phi_{\text{odd}} \), respectively the subsets of vertices of \( \Phi \) of even and odd weight. Let \( \tau_{\text{even}} \) be the union of \( \Phi_{\text{even}} \), \( \tau_1(\Phi_{\text{odd}}), \tau_2(\Phi_{\text{odd}}), \ldots, \tau_{n-1}(\Phi_{\text{odd}}) \). Let \( \tau_{\text{odd}} \) be the union of \( \Phi_{\text{odd}}, \tau_1(\Phi_{\text{even}}), \tau_2(\Phi_{\text{even}}), \ldots, \tau_{n-1}(\Phi_{\text{even}}) \).

Let \( \tau \) be \( ((m_{n})^{-1}(\tau_{\text{even}})Q^0(n-1)) \cup ((m_{n})^{-1}(\tau_{\text{odd}})Q^0(n-1)) \). Then \( \tau \) is the union of sets \( V_i \) for \( i = 1, \ldots, n \) defined as follows: \( V_n \) is the union of the copy of \( \Phi_{\text{odd}} \) in \( Q_n \) and the copy of \( \Phi_{\text{even}} \) in \( Q_n \); \( V_i \) is the union of the copy of \( \tau_i(\Phi_{\text{even}}) \) in \( Q_n \) and the copy of \( u_i(\Phi_{\text{odd}}) \) in \( Q_n \), for \( i = 1, 2, \ldots, n-1 \).

Clearly, the set \( \text{Odd}_n \) of all vertices of \( Q_n \) of odd weight equals the union of the \( V_i \), for \( i = 1, \ldots, n \). Moreover, each vertex in \( Q_n - \text{Odd}_n \) is a neighbor of \( V_i \), for \( i = 0, 1, \ldots, n-1 \). If we had at our disposal matchings \( E_i \), for \( i = 1, \ldots, n \) as above described, we would get immediately a twisted LPDS. We know how to do this only for \( n = 4 \) (Felzenbaum's) and for \( n = 8 \). This last case is explained in Section 5.

4. An LPDS with components in the Eight Parallel 1-Factors of \( Q_n \).

A perfect Hamming code \( \Phi \) in \( Q_7 \) can be described in terms of a
Steiner triple system associated to the Fano projective plane. The set of elements of the Fano plane is taken as the set \( I_7 = \{1,2,...,7\} \).
The Steiner triples, representing the lines of the Fano plane, can be taken then cyclically as \( \mu(1)=235, \mu(2)=346, \mu(3)=457, \mu(4)=561, \mu(5)=672, \mu(6)=713, \mu(7)=124 \). We identify \( I_7 \) with the cyclic group \( \mathbb{Z}_7 \), where 7 is identified with 0 and the other symbols are the same, so we have now \( \mu(i) = \{i+1,i+2,i+4\} \), for \( i \in \mathbb{Z}_7 \), a cyclic presentation of the Steiner triples. Now notice that the given correspondence \( \mu \) has the following nice property \( P_\mu \): 
\[
P_\mu: \text{If } \mu(i) = \{j,k,l\}, \text{ then } i \notin \mu(i) \text{ and } i \in \mu(p) \text{ for each } p \in I_7 - \{(i)\}.
\]
Moreover, this same property is shared by the correspondence \( \tau \) given for each \( i \in I_7 \) by \( \tau(i) = I_7 - \{(i)\} \cup \mu(i) \) (or by \( \tau(i) = \{i-1,i-2,i-4\} \) for \( i \in \mathbb{Z}_7 \), as property \( P_\tau \):
\[
P_\tau: \text{If } \tau(i) = \{j,k,l\}, \text{ then } i \notin \tau(i) \text{ and } i \in \tau(p) \text{ for each } p \in I_7 - \{(i)\} \cup \mu(i) \).
\]
Observe that $\mu(i) = \tau(-i)$ for every $i \in I_7$, so $\tau$ yields also a cyclic presentation of Steiner triples (opposite in sign to those offered by $\mu$). Notice that $\mu_i'$ and $\mu_i''$ can now be expressed as:

$\mu_i'$: If $\mu(i) = \{j, k, l\}$, then $i \notin \mu(p)$ for each $p \in \mu(l) = \tau(-i)$.

$\mu_i''$: If $\tau(l) = \{j, k, l\}$, then $i \notin \tau(p)$ for each $p \in \mu(l) = \tau(-i)$.

It is known that each three columns of the matrix $H$ of Section 3 which are linearly dependent correspond to a Steiner triple. In order to have the code $\Phi$ in $G_7$ selected as to fit the cyclic presentation $\mu$ of a Steiner triple system above, we take for example the columns $1, 2, 3, 4, 5, 6, 7$ of $A$ to be respectively the binary expressions of the integers $4, 2, 1, 6, 3, 7, 5$. This is used below and in Section 5.

Inspired by the correspondences $\mu$ and $\tau$ and by the properties $\mu_i'$ and $\mu_i''$, we were able to produce the following twisted LPDS in $G_8$ having two edges in each coordinate direction, expressed by means of both Rado's and the vector notation for the vertices of the 8-cube and our own initiating notation for the edges of $G_8$, where the last eight edges presented are opposite in $G_8$ to the first eight ones:

$G_8 = (0, 8) = (00000000, 00000001);
235 = (235, 2351) = (01101000, 11101000);
346 = (346, 3462) = (00110100, 01110100);
457 = (457, 4573) = (00011010, 00111010);
561 = (561, 5614) = (10001100, 10011100);
672 = (672, 6725) = (01000110, 01001110);
713 = (713, 7136) = (10100010, 10100110);
124 = (124, 1247) = (11010000, 11010010);
12345678 = (1234567, 12345678) = (11111110, 11111111);
467 = (4678, 46781) = (00010111, 10010111);
571 = (5718, 57182) = (10001011, 11001011);
6128 = (6128, 61283) = (11000101, 11100101);
7238 = (7238, 72384) = (01100011, 01110011);
1348 = (1348, 13485) = (10110001, 10111001);
2458 = (2458, 24586) = (01011001, 01011101);
3568 = (3568, 35687) = (00011001, 00110111);

5. The Case $2n = 16$.

It was found in [1,2] that $Q_7 - \Phi_0$ is the edge-disjoint union of two edge-transitive (but not vertex-transitive) isomorphic spanning trivalent subgraphs of girth 10 which are 8-coverings of the Heawood graph (the incidence graph of the Fano plane). To represent one of these
two subgraphs, to be called from now on the red graph and the blue graph (though this coloring will be explained later on) let us consider the Heawood graph $H$ labelled as in Figure 2, where each vertex is labelled with a sign and one of the numbers from 1 to 7, and each edge is labelled with one of the numbers from 1 to 7. Observe that each edge has an endvertex with each sign and that its labelling and those of its endvertices form a Steiner triple as given by $\mu$ above.

The perfect code $\Phi$ will be given with the following notation:

$$+0 = 1111111; -0 = 0000000; \text{for each } i \text{ from } 1 \text{ to } 7, \text{ let } -i \text{ (respectively } +i) \text{ stand for the } 7\text{-tuple obtained from } +0 \text{ (respectively } -0) \text{ by complementing the coordinates corresponding to the components of } \mu(i).$$

The neighbors of $\Phi$ in $Q_7$ will be described with the following notation:

$$j(\delta i), \text{ where } \delta \text{ is a sign; } i \text{ is a number from } 0 \text{ to } 7, \text{ so that } \delta i \text{ represents a vertex of } \Phi; j \text{ is a number from } 1 \text{ to } 7; \text{ and we take } j(\delta i) \text{ as obtained from } \delta i \text{ by complementing its } j^\text{th} \text{ coordinate.}$$

Now we define the red subgraph $R$ of $Q_7$ mentioned above as an 8-covering graph of $H$: If $\sigma: R \rightarrow H$ is the corresponding covering map, then the inverse image $\sigma^{-1}(\delta j)$ of a typical element $\delta j$ of $H$, where $\delta$ is a sign $+$ or $-$ and $j \in \{1, \ldots, 7\}$, is composed by those vertices of $Q_7$ obtained from $\delta i$ by complementing the $j^\text{th}$ coordinate, for $j$ running from 0 to 7.

Let $e$ be an edge of $H$ with label $y(e)$ and endvertices $u$ and $v$ labelled respectively with $+y(u)$ and $-y(v)$. The 16 vertices of $\Phi$ can be partitioned into eight pairs $(+a_i, -b_i)$, $(i=0, \ldots, 7)$, where distance $(+a_i, -b_i) = 3$ by means exactly of edges in coordinate directions $y(u)$, $y(e)$ and $y(v)$. There are eight corresponding 3-subcubes $Q_i = Q_i(y(u), y(e), y(v))$ of $Q_7$ using exactly edges in these coordinate directions and such that $Q_i$ contains $+a_i$ and $-b_i$. Let $C_i$ be the 6-cycle obtained from $Q_i$ by deletion of $(+a_i, -b_i)$, $(i=0, \ldots, 7)$. Then $\sigma^{-1}(e)$ is composed by eight edges in the direction $y(e)$, one per 6-cycle $C_i$.

The advantage of the present notation with respect to that of [2] is that there are canonical 6-cycles in $H$ whose inverse images via $\sigma$ are composed each by four canonical 12-cycles of $R$. With the present notation of Figure 2, we have these cycles cyclically settled. This is seen if we describe these 6-cycles in Figure 2 by expressing the successive occurrence of vertex and edge labels:

$$(-i+1, i, +3+i, 5+i; -(2+i), i, +(6+i), 3+i, -(4+i), i, +(5+i), 6+i),$$

where $i=1, \ldots, 7$ and labels are taken modulo 7, with 7 replacing 0. Notice that each of these 6-cycles has alternate edges labelled $i$ that with the other edge labels in it, namely $5+i, 3+i$ and $6+i$, leaves...
out the components of a specific Steiner triple: \((1+i, 2+i, 4+i)\).

Now, \(H\) admits a 1-factorization \(F\) with these particular properties:

1. Each two 1-factors of \(F\) form a Hamilton cycle; (this property is rare, however shared by the graph of the regular dodecahedron);
2. Each 1-factor of \(F\) has exactly one edge in each label, from 1 to 7.

These 1-factors are shown in Figure 2 by tracing differently the edges of each one of them in particular.

Now select a specific 1-factor \(f\) of \(F\). The inverse image through \(\sigma\) of an edge \(e\) of \(f\) is composed by eight parallel edges of \(Q_7\). We want to add an eighth coordinate to each one of the edges of \(\sigma^{-1}(e)\) so that the resulting eight edges of \(Q_8\) and those obtained from them by complementation take the place of an \(E_i\) as in Section 3.

We transform each edge \(h\) of \(\sigma^{-1}(e)\) in \(Q_7\) into an edge \(h'\) of \(Q_8\) by considering the value \(s\) of the \(y(e)\text{th}\)-coordinate of the odd-weighted endvertex of \(h\) and by setting \(s\) as the value of the eighth coordinate of the endvertices of \(h'\), being the remaining coordinate values the same as in \(h\). The resulting eight edges of \(Q_8\) and those other eight edges obtained from these by complementation constitutes an \(E_i\) as desired at the end of Section 3.

By doing this for each one of the edges of \(f\), we get the edge sets \(E_i\) for \(i = 1, \ldots, 7\). The edge set \(E_0\) is \((m_0)^{-1}(\emptyset)\).

We know that there are \(2^7\) vertices in \(Q_8\) with odd weight. Section 3 provides us with a partition of these \(2^7\) vertices into eight subsets \(V_0, V_1, \ldots, V_7\). Thus each vertex subset \(V_i\) in the big \(Q_8\), has \(2^{7/3} = 2^4\) vertices for \(i = 0, 1, \ldots, 7\). As in Section 3, in the copy of the small 8-cube represented by each vertex of \(V_i\), we consider the corresponding copy of \(E_i\), containing \(2^4 = 16\) edges, as contributing to the totality of our desired LPDS. This yields \(2^8 = 2^4.2^4\) edges parallel to each one of the last eight coordinate directions of \(Q_{16}\), a totality of \(2^8\) edges in \(Q_{16}\), as claimed in (1) of the Introduction.

In the description just given of an LPDS in \(Q_{16}\) using eight coordinate directions, we were able to avoid citing directly the projections \(m_i\) suggested in Section 3, but for the last projection \(m_8\), explicitly defined as \(m_8\) in that section.
Remark on Nonlinear Error-Correcting Codes.

V. Zinoviev called our attention to the existence of nonlinear error-correcting codes in the cubes $Q_n$ with $n = 2^k - 1$ and $k > 3$, including those leading to isolated PDS's. Two constructions leading to these are mentioned in [6] pages 77 (Vasiliev's) and 591 (Zinoviev's). Thus, we have at our disposal not only the perfect Hamming codes, which are linear, but also these other codes, in order to produce isolated PDS's.

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