Ternary Hamming and Binary Perfect Covering Codes

Italo J. Dejter and Kevin T. Phelps

Abstract. We establish that the perfect dominating set of the 13-cube found by Östergård and Weakley (as a counterexample to Weichsel’s uniformity conjecture) exists very naturally in the ternary Hamming code of length 13 and has a very special geometric-combinatorial structure. This approach provides very short proofs of its characteristic properties.

1. Introduction

Given \(1 < n \in \mathbb{Z}\), the \(n\)-cube \(Q_n\) is the graph with vertex set \(V = \{0, 1\}^n\) and edge set \(E = \bigcup_{i=0}^{n-1} f^i\), where \(f^i = \{(v, w) \in V \times V : (v - w)_j = 1\) if and only if \(j = i, \{j = 0, \ldots, n-1\}\), for \(i = 0, \ldots, n-1\). The edges of \(f^i\) are said to have (or to lie along) the direction \(i\), for \(i = 0, \ldots, n-1\). Given a graph \(G\), we say that \(S \subseteq V(G)\) is a perfect dominating set (or PDS) of \(G\) if every \(v \in V(G) \setminus S\) is neighbor of exactly one vertex of \(S\). Weichsel [6] showed that each connected component of the subgraph induced by a PDS in \(Q_n\) is a subcube and conjectured that \(S\) is uniform, i.e. all such components are isomorphic. Östergård and Weakley [3, 4] showed that the conjecture is false, by exhibiting a PDS \(S\) in \(Q_{13}\) whose induced subgraph has as connected components 26 4-cubes and 288 isolated vertices (which correspond to the words with no zero coordinates of the ternary Hamming code) and whose automorphism group is the general linear group \(GL(3, 3)\).

Perfect dominating sets satisfy an analog of the sphere-packing bound and can be thought of as generalizations of perfect codes. Let \(S \subseteq V\) be a PDS in \(Q_n\). Recall that each component of \(S\) is an \(i\)-cube, where \(0 \leq i \leq n\), [6]. For such a fixed \(i\), let \(X_i\) be the number of \(i\)-cubes which are components of \(S\). A sphere of radius 1 about any vertex in such an \(i\)-cube will cover (or "dominate") exactly \(n - i\) vertices not in this \(i\)-cube. For \(S\) to be a PDS, the spheres must cover every vertex of \(Q_n\) and their intersections with \(Q_n \setminus S\) must be disjoint. For these intersections to be disjoint the Hamming distance between any two vertices in different components of \(S\) must be at least 3. Equivalently, the distance between any two components of \(S\) is at least 3. Equivalently, no vertex of \(Q_n \setminus S\) can be dominated twice, that is by two different vertices of \(S\).

Definition 1.1. Perfect Domination Bound.

1991 Mathematics Subject Classification. Primary 94B05; Secondary 05C99.

© 2001 American Mathematical Society
If no vertex of \( Q_n \setminus S \) can be dominated twice, then
\[
\sum_{i=0}^{i=n} 2^i(n - i + 1)X_i \leq 2^n.
\]

**Proposition 1.1.** A set \( S \) of vertices in the \( n \)-cube is a PDS if and only if
the distance between any two components is at least 3 and the number of \( i \)-cube components satisfies the Perfect Domination Bound with equality.

The proof is elementary and left to the reader.

**Remark:** Recall that Weischel [6] showed that a necessary condition for a perfect dominating set in the \( n \)-cube to have an \( i \)-cube as a component was that \( n - i \equiv 1 \) or 3 (mod 6). In other words, in the Perfect Domination Bound above \( X_i \) must be zero whenever this numerical condition is not met. This puts a significant limitation on the set of numerically feasible solutions for a PDS. This may help for example in the tasks suggested in our Conclusions below.

The \( i \)-cube components of a PDS can be represented by codewords over a ternary alphabet. The vertices of an \( i \)-cube component all agree on \( n - i \) coordinates. Take any vertex in such a component and put another symbol, *, in the \( i \) coordinates which form the \( i \)-cube. We then have one codeword in \( \{\ast, 0, 1\} \) for each component. The distance required between ternary codewords is not the standard Hamming distance however.

**Definition 1.2.** \((0, 1, \ast)\)-Distance
\[
D_{(0,1,\ast)}(x, y) = \{|j; \{x_j, y_j\} = \{0, 1\}\}.
\]

The name \((0, 1, \ast)\)-Distance arises for example in [5]. The \((0, 1, \ast)\)-distance differs from the Hamming distance \( HD(x, y) \) in that we ignore disagreements which involve the symbol \( \ast \). Thus,
\[
D_{(0,1,\ast)}(x, y) \leq HD(x, y).
\]

Now we think of PDS's in terms of ternary codes with minimum Hamming distance at least 3. Instead of the alphabet \( \{\ast, 0, 1\} \) we will use the more natural ternary alphabet \( \{0, 1, 2\} \) with 0,1,2 taking the place of \( \ast,0,1 \), respectively. We show that the Ostergård-Weakley PDS \( S \) lives very naturally inside the ternary Hamming Code of length 13 and has a very special geometric-combinatorial structure.

### 2. Ternary Hamming Code

As we remarked in the Introduction, the problem of finding perfect dominating sets can be reduced to finding appropriate ternary codes. For length \( n = 13 \), one feasible solution to the Perfect Domination Bound would involve 26 codewords of weight 9 and 288 codewords of weight 13. It is natural to look for such a ternary code inside the ternary Hamming code of length 13. Information about ternary Hamming codes can be found in MacWilliams and Sloane, [2], chapters 5, 6.

The support of a codeword is the set of nonzero coordinates of the codeword. If two codewords in a ternary code have the same support, then their Hamming distance is the \((0,1,\ast)\)-distance. There are 288 codewords of weight 13 in the ternary Hamming code and obviously the minimum distance is at least 3. (Ostergård had previously observed this.) Denote this set of codewords by \( S_{13} \).
The dual of the Hamming code has 27 words: the zero word and 26 words of weight 9. The sum and difference of any two codewords is again a codeword and thus has weight 9 (or 0).

Consider the words of weight 9 in the dual of the Hamming code (which is a sub-code of the Hamming code because the rows of a parity check matrix are orthogonal also to themselves). Take any such word without loss of generality (000111111111). A word that is not a multiple must have the form (0111000111222) because the two words can be rows of a parity check matrix which has different projective points as columns. So, any two such words of weight 9 have (0,1,*)-distance 3. A word of weight 13 must be orthogonal to (000111111111). If there are not three 2's among the final nine coordinates, these must all be 1. Then the initial part (a,b,c,d) is orthogonal to (0111) and (1012) and hence does not have weight 4, a contradiction. So, a word of weight 13 has (0,1,*)-distance at least 3 to a word of weight 9.

**Theorem 2.1.** The ternary code $S$ consisting of the 288 words of weight 13 in the ternary Hamming code and the 26 words of weight 9 in the dual code, forms a perfect dominating set.

The automorphism group of the ternary code $S$ is the same as that of the ternary Hamming code since any automorphism must be an automorphism of the dual code $(S_9)$, and the words of weight 13 $(S_{13})$.

3. Conclusion

The optimal perfect dominating set in the 4-cube exists as a subcode of the ternary Hamming code of length 4. With the result of this paper, it is tempting to conjecture that the PDS ternary codes of length $n = (3^r - 1)/(3 - 1)$ can always be found inside the ternary Hamming code. Before venturing into such a conjecture it would be worthwhile to first establish this result for $r = 4$ and $r = 5$, ($n = 40$ and $n = 121$).

Of course, the Remark after Proposition 1.1 must be taken into account for these purposes.

**Acknowledgements.** P. R. J. Östergård has communicated initially that there are 288 codewords of weight 13 in the ternary Hamming code. We are also indebted to our anonymous referee for his helpful corrections and suggested improvements.

**References**


