From the Coxeter graph to the Klein graph

Italo J. Dejter
University of Puerto Rico
Rio Piedras, PR 00936-8377
ijdjter@uprrp.edu

Abstract

We show that the 56-vertex Klein cubic graph $\Gamma'$ can be obtained from the 28-vertex Coxeter cubic graph $\Gamma$ by 'zipping' adequately the squares of the 24 7-cycles of $\Gamma$ endowed with an orientation obtained by considering $\Gamma$ as a $C$-ultrahomogeneous digraph, where $C$ is the collection formed by both the oriented 7-cycles $\vec{C}_7$ and the 2-arcs $\vec{P}_3$ that tightly fasten those $\vec{C}_7$ in $\Gamma$. In the process, it is seen that $\Gamma'$ is a $C'$-ultrahomogeneous (undirected) graph, where $C'$ is the collection formed by both the 7-cycles $C_7$ and the 1-paths $P_2$ that tightly fasten those $C_7$ in $\Gamma'$. This yields an embedding of $\Gamma'$ into a 3-torus $T_3$ which forms the Klein map of Coxeter notation $(7,3)_8$. The dual graph of $\Gamma'$ in $T_3$ is the distance-regular Klein quartic graph, with corresponding dual map of Coxeter notation $(3,7)_8$.

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1 Introduction

The study of ultrahomogeneous graphs (resp. digraphs) can be traced back to [20, 12, 19, 5, 14], (resp. [11, 16, 6]). Following a line of research initiated in [15], given a collection $C$ of (di)graphs closed under isomorphisms, a (di)graph $G$ is said to be $C$-ultrahomogeneous (or $C$-UH) if every isomorphism between two induced members of $C$ in $G$ extends to an automorphism of $G$. If $C = \{H\}$ is the isomorphism class of a (di)graph $H$, we say that such a $G$ is $\{H\}$-UH or $H$-UH. In [15], $C$-UH graphs are defined and studied when $C$ is the collection of either (a) the complete graphs, or (b) the disjoint unions of complete graphs, or (c) the complements of those unions.

We may consider a graph $G$ as a digraph by considering each edge $e$ of $G$ as a pair of oppositely oriented (or O-O) arcs $\vec{e}$ and $(\vec{e})^{-1}$. Then, zipping or fastening $\vec{e}$ and $(\vec{e})^{-1}$, operation that we define as uniting $\vec{e}$ and $(\vec{e})^{-1}$, allows to obtain precisely $e$, a simple technique to be used below. (In [10], however, a strongly connected $C_4$-UH oriented graph without O-O arcs was presented). In other words, $G$ must be a graph considered as a digraph, that is, for any two vertices
having exactly $\eta$ cubic graph of order $F$.

The Coxeter graph $\Gamma = \{\{I\}\}_{M-UH}$ graph if, for each copy $H_0$ of $H$ in $G$ containing a copy $M_0$ of $M$, there exists exactly one copy $H_1 \neq H_0$ of $H$ in $G$ with $\{V(H_0)\cap V(H_1) = V(M_0)\}$ and $\{E(H_0)\cap E(H_1) = E(M_0)\}$. The vertex and edge conditions above can be condensed as $H_0 \cap H_1 = M_0$. We may say that such a $G$ is tightly fastened, which can be generalized by saying that an $\{H\}_{M-UH}$ graph $G$ is an $\ell$-fastened $\{H\}_{M-UH}$ graph if given a copy $H_0$ of $H$ in $G$ containing a copy $M_0$ of $M$, then there exist exactly $\ell$ copies $H_i \neq H_0$ of $H$ in $G$ such that $H_i \cap H_0 \supseteq M_0$, for each $i = 1, 2, \ldots, \ell$, with at least $H_1 \cap H_0 = M_0$.

Now, let $\bar{M}$ be a subdigraph of a digraph $\bar{H}$ and let the graph $G$ be both an $\bar{M}$-UH and an $\bar{H}$-UH digraph. We say that $G$ is an $\{\bar{M}\}_{\bar{H}}$-UH digraph if for each copy $\bar{H}_0$ of $\bar{H}$ in $G$ containing a copy $\bar{M}_0$ of $\bar{M}$ there exists exactly one copy $\bar{H}_1 \neq \bar{H}_0$ of $\bar{H}$ in $G$ with $\{V(\bar{H}_0)\cap V(\bar{H}_1) = V(\bar{M}_0)\}$ and $\{A(\bar{H}_0)\cap A(\bar{H}_1) = A(\bar{M}_0)\}$, where $A(\bar{H}_1)$ is formed by those arcs $(\bar{e})^{-1}$ whose orientations are reversed with respect to the orientations of the arcs $\bar{e}$ of $A(\bar{H}_1)$. Again, we may say that such a $G$ is tightly fastened. This case is used in the construction of Section 3.

The Coxeter graph $\Gamma = F_{028}A$ [2] is a distance-transitive hypohamiltonian [1] cubic graph of order $n = 28$, diameter $d = 4$, girth $g = 7$, arc-transitivity $k = 3$, having exactly $\eta = 24$ $g$-cycles, $a = 336$ automorphisms, intersection array $I = \{3, 2, 2, 1; 1, 1, 1\}$ and weakly regular parameters $W = \{(28, 3), (0, 0, 1)\}$.

The Klein cubic graph $\Gamma' = F_{056}B$ is a hamiltonian cubic graph with $n' = 2n$, $d' = 6$, $g' = g$, $k' = 2$, $\eta' = \eta$, $d' = a$ and $W' = \{(24, 7), (2), (0, 2)\}$, (not to be confused with the bipartite double graph of $\Gamma$, denoted $F_{056}C$); see [2, 21, 18, 17]. (We remark that $\Gamma$ can be obtained as the graph whose vertices are the 6-cycles of the Heawood graph $\Gamma'' = F_{014}A$ [2], with any two vertices adjacent if and only if the 6-cycles they represent are disjoint, where we recall that $\Gamma''$ is a distance-transitive hamiltonian cubic graph with $n'' = 14$, $d'' = 3$, $g'' = 6$, $k'' = 4$, $\eta'' = n$, $a'' = a$, $I'' = \{3, 2, 2; 1, 1, 3\}$ and $W'' = \{(14, 3), (0), (0, 1)\}$.)

Given a finite graph $H$ and a subgraph $M$ of $H$ with $|V(H)| > 3$, we say that a graph $G$ is (strongly fastened) or SF $\{(H)\}_{M-UH}$ if there is a sequence of connected subgraphs $M = M_1, M_2, \ldots, M_t \subseteq K_2$ such that: (a) $M_{i+1}$ is obtained from $M_i$ by the deletion of a vertex, for $i = 1, \ldots, t-1$ and (b) $G$ is a $(2^t - 1)$-fastened $\{H\}_{M-UH}$ graph, for $i = 1, \ldots, t$. Theorem 1 below asserts that $\Gamma$ is an SF $\{C_7\}_{E_5}$-UH graph.

Theorem 2 establishes that $\Gamma$ is a $\{C_7\}_{E_5}$-UH digraph. In Section 3, squaring the resulting oriented 7-cycles allows the recovery of $\Gamma'$ dressed up as a $\{C_7\}_{E_5}$-UH graph, via zipping of the O-O induced 2-arcs shared (as 2-paths) by the pairs of O-O 7-cycles.
As in [2, 21, 18, 17], the dual graph of $\Gamma'$ with respect to an embedding of its 24 7-cycles into a 3-torus (known as the Klein map, of Coxeter notation $(7,3)_8$, see argument previous to Theorem 3, below) is the Klein quartic graph $\mathcal{K}$ (of Corollary 4), a 24-vertex distance-regular graph with intersection array $\{7, 4, 1; 1, 2, 7\}$ and weakly regular parameters $(24, (7), (2), (0, 2))$.

2 \quad \{C_7\}_{P_3} \text{-UH and } \{\tilde{C}_7\}_{\tilde{P}_3} \text{-UH properties of } \Gamma$

**Theorem 1** $\Gamma$ is an SF $\{C_g\}_{P_{i+2}} \text{-UH graph, for } i = 0, 1$. In particular, $\Gamma$ is a $\{C_7\}_{P_3} \text{-UH graph and has exactly } 6n^{-1} = 24$ $g$-cycles.

**Proof.** We have to see that $\Gamma$ is a $(2^{i+1} - 1)$-fastened $\{C_g\}_{P_i} \text{-UH graph, for } i = 0, 1$. In fact, each $(2 - i)$-path $P = P_{i-1}$ of $\Gamma$ is shared exactly by $2^{i+1}$ $g$-cycles of $\Gamma$, for $i = 0, 1$. This and a simple counting argument for the number of $g$-cycles yield the assertions in the statement.

In fact, the proof above can be extended in order to establish that every distance-transitive cubic graph $G$ with girth $= g$ and AT $= k$, (including $G = \Gamma''$), is an SF $\{C_g\}_{P_{i+2}} \text{-UH graph, for } i = 0, 1, \ldots, k - 2$, and in particular a $\{C_g\}_{P_k} \text{-UH graph with exactly } 2^{k-2}3n^{-1}g$-cycles.

Given a $\{C_g\}_{\tilde{P}_k} \text{-UH graph } G$, an assignment of an orientation to each $g$-cycle of $G$ such that the two $g$-cycles shared by each $(k - 1)$-path receive opposite orientations yields a $\{C_g\}_{\tilde{P}_k} \text{-orientation assignment, (or } \{C_g\}_{\tilde{P}_k} \text{-OA). The collection of } \eta \text{ oriented } g\text{-cycles corresponding to the } \eta \text{ g-cycles of } G, \text{ for a particular } \{C_g\}_{\tilde{P}_k} \text{-OA will be called an } \{\eta C_g\}_{\tilde{P}_k} \text{-OAC. Each such cycle will be expressed with its successive composing vertices expressed between parentheses but without separating commas, (as is the case for arcs } (u, v) \text{ and 2-arcs } (u, v, w), \text{ where as usual the vertex that succeeds the last vertex of the cycle is its first vertex.}$

**Theorem 2** $\Gamma$ is $\{C_g\}_{\tilde{P}_k} \text{-UH, or } \{C_7\}_{\tilde{P}_3} \text{-UH.}$

**Proof.** $\Gamma$ is obtained from three 7-cycles $(u_1u_2u_3u_4u_5u_6u_7)$, $(v_4v_5v_6v_7v_8v_9v_{10})$, $(t_3t_4t_5t_6t_7t_8t_9)$ by adding a copy of $K_{1,3}$ with degree-1 vertices $u_x, v_x, t_x$ and a central degree-3 vertex $z_x$, for each $x \in Z_7$. Then $\Gamma$ admits the $\{24 \tilde{C}_7\}_{\tilde{P}_3} \text{-OAC:}$

$$\begin{align*}
\{P\} &:= (u_1u_2u_3u_4u_5u_6u_7), \\
\{Q\} &:= (v_4v_5v_6v_7v_8v_9v_{10}), \\
\{R\} &:= (t_3t_4t_5t_6t_7t_8t_9), \\
\{G\} &:= (u_1u_2u_3u_4u_5u_6u_7), \\
\{H\} &:= (v_4v_5v_6v_7v_8v_9v_{10}), \\
\{I\} &:= (t_3t_4t_5t_6t_7t_8t_9).
\end{align*}$$

In fact, Theorem 2 can be adapted to a statement for every distance-transitive cubic graph which is neither $\Gamma''$ nor the Petersen, Pappus or Foster graphs.
3 ‘Zipping’ the squares \((\vec{C}_7)^2\) in \(\Gamma\) towards \(\Gamma’\)

In this section, we keep using the construction and notation of \(\Gamma\) and of its \(\{24\vec{C}_7\}_{\vec{P}_0}\)-OAC, as conceived in the proof of Theorem 2. Consider the collection \((\vec{C}_7)^2(\Gamma)\) of squares of oriented 7-cycles in the \(\{24\vec{C}_7\}_{\vec{P}_0}\)-OAC of \(\Gamma\) in that proof. From now on, each initial vertex \(w_0\) of an arc \(\vec{e} = (w_0, w_1)\) of a member \(\vec{C}_7^2\) of \((\vec{C}_7)^2(\Gamma)\), the arc \(\vec{e}\) itself and its terminal vertex \(w_1\) are respectively indicated by, or marked with, the symbols \(v\) indicated exactly by means of those same symbols, namely \(u\).

Next, we explain how this operation \(\Gamma \to \{24\vec{C}_7\}_{\vec{P}_0}\)-OAC(\(\Gamma\)) is composed. The Fano plane \(\mathcal{F}\), with point set \(J_7 = \{1, \ldots, 7\}\) and line set \(\{124, 235, 346, 457, 561, 672, 713\}\), yields a coloring of the vertices and edges of \(\Gamma\), as represented on the upper left quarter of Figure 1, below, where the color of each vertex \(v\) of \(\Gamma\) (written in boldface in the next paragraph, for clarity) and the colors of its three incident edges form a quadruple \(q\) whose complement \(\mathcal{F} \setminus q\) is used to denote \(v\), ([13] page 69). Moreover: (a) the triple formed by the colors of the edges incident to each \(v\) of \(\Gamma\) is a line of \(\mathcal{F}\); (b) the color of each edge \(e\) of \(\Gamma\) together with the colors of the endvertices of \(e\) form a line of \(\mathcal{F}\).

The vertices \(u_x, z_x, v_x, t_x\) created in the presentation of the \(\{24\vec{C}_7\}_{\vec{P}_0}\)-OAC in the proof of Theorem 2 are depicted concentrically in the mentioned representation of \(\Gamma\) in Figure 1, from the outside in, starting say downward from top with colors \(x = 1, 5, 4, 3\) for respective vertices \(257 = \mathcal{F} \setminus 1364, 134 = \mathcal{F} \setminus 5602, 567 = \mathcal{F} \setminus 4013, 356 = \mathcal{F} \setminus 3214\), which are shown solid in the figure against a backdrop of the remaining hollow vertices.

The squares \(\vec{C}_7^2\) corresponding to the 24 oriented 7-cycles \(\vec{C}_7\) of \(\Gamma\) are represented: (a) via their induced cyclically-presented orientations and (b) with each vertex \(v\) (resp. arc \(\vec{e}\)) of a \(\vec{C}_7^2\) conveniently indicated by means of a color \(c(v)\) for \(v\) (resp., conveniently indicated by means of a subindex, color \(c(u)\) for
the middle vertex \( u \) of the 2-path \( \bar{E} \) of \( \overrightarrow{C}_7 \) that \( \bar{e} \) represents). The net effect that this color notation produces makes the 24 oriented 7-cycles \( \overrightarrow{C}_7^2 \) pairwise distinguishable, thus providing them with a distinctive and well-defined presentation. As an example, we go back to the oriented 7-cycle \( \overrightarrow{C}_7^2 = ([1])^2 \) pictured above, showing now how it receives its colors \( c(u_i) \):

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \hline
1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}
\]

which can be written in short as \( (12350712456) \), meaning that \( c(u_0) = 7 \), \( c(u_i) = i \), for \( i = 1, \ldots, 6 \) and if \( \bar{e} = (u_i, u_{i+2}) \), with \( i + j \) taken mod 7 for \( j = 1, 2 \), where 0 is rewritten as 7, then \( c(\bar{e}) = i + 1 \), this color written as a subindex.

![Figure 1: \( \mathcal{F} \)-colored \( \Gamma \) and the three charts of \( \Gamma' \)](image)

Continuing this way, the oriented 7-cycles \( \overrightarrow{C}_7^2 \), indicated by means of the symbols \( i^j \) (corresponding respectively to their square-root cycles \( \overrightarrow{C}_7 = \{i\}^2 \)), where \( i \in \{0\} \cup J_7 \) and \( j \in J_3 = \{1, 2, 3\} \), are presented now as follows, by means of the colors \( c(u_i) \) for their composing vertices and arcs, that make them pairwise distinguishable, as claimed, thus providing a distinctive and well-defined
Each 2-arc of $\Gamma$ is suggested exactly once in these oriented cycles $i^j$. Each 2-path of $\Gamma$ is suggested twice in them, once for each one of its two composing O-O 2-arcs. The assumed orientation of each $C^2_i = i^j$ corresponds with, and is induced by, the orientation of the corresponding 7-cycle $C_7^i = i^j$.

Each 2-path $E$ of $\Gamma$ separates two of its 24 7-cycles, say $\vec{e}_j$ and $\vec{k}_j$, with opposite orientations over $E$. Now, these $\vec{e}_j$ and $\vec{k}_j$ restrict to the two different 2-arcs provided by $E$, say 2-arcs $\vec{E}$ and $(\vec{E})^{-1}$. Then, $\vec{E}$ and $(\vec{E})^{-1}$ represent corresponding arcs $\vec{e}$ and $(\vec{e})^{-1}$ in $i^j$ and $k^j$, respectively.

Let us see that $\vec{e}$ and $(\vec{e})^{-1}$ can be zipped into an edge $e$ of $\Gamma'$. In fact, $\Gamma'$ can be assembled from the three charts shown on the upper right and bottom of Figure 1 by zipping the oriented 7-cycles $i^j$, interpreted all with counterclockwise orientation. Each of these three charts conforms a ‘rosette’, where the oriented 7-cycles $i^j$ with $i \neq 0$ are represented as ‘petals’ of the ‘central’ oriented 7-cycles $0^1$, $0^2$ and $0^3$. (Similarly, the assembly of $\Gamma'$ could have been done also around $i^1$, $i^2$ and $i^3$, taken as ‘central’ oriented 7-cycles, for any $0 \neq i \in J_7$).

Moreover, each arc $\vec{e}$ in the external border of any selected one of the three charts, ($\vec{e}$ interpreted as an arc of an oriented cycle $C^2_i$ in the selected chart), is accompanied, externally to the chart, by the symbol $i^j$ of another oriented 7-cycle $i^j$ that also contains $\vec{e}$ and forms a ‘petal’ in just one of the other two (‘rosette’) charts. For example, the oriented cycle $3^1$ on the left of the chart centered at the oriented 7-cycle $0^1$ (on the upper-right of Figure 1) has its leftmost arc $\vec{e}$, corresponding to the symbol subsequence $623$ in $3^1 = (134621347652)$, also present in reverse in the oriented cycle $1^3 = (1745631472563)$, that is to say as $(\vec{e})^{-1}$, corresponding to the symbol subsequence $256$, at the upper-right in the chart centered at the oriented 7-cycle $0^3$ (on the lower-right of Figure 1). Thus, the symbols $13$ and $31$ accompany the representation of the arcs $\vec{e}$ and $(\vec{e})^{-1}$ on the outside of the external borders of their respective charts. Not only the symbol $3^1$ indicates externally the arc $(\vec{e})^{-1}$ of the oriented 7-cycle $1^3$, but also indicates an arc $\vec{f}$ of the oriented 7-cycle $5^3$, the one corresponding to the symbol subsequence $674$ in $5^3 = (13674251367425)$. The arc $(\vec{f})^{-1}$ is in the first mentioned oriented 7-cycle, $3^1$, just up from $\vec{e}$ and preceding it in the 28-cycle delimiting externally the chart centered at the 7-cycle $0^1$, with corresponding symbol subsequence $465$ in $3^1 = (1347621347652)$.

The presence of these arcs, $\vec{e}$, $(\vec{e})^{-1}$, $\vec{f}$ and $(\vec{f})^{-1}$, (and in all other similar cases) is expressed in the following formulation of the three oriented 28-cycles
are 8-cycles).\}
\{i \in T \text{ (for having opposite orientations makes them mutually cancelable), } \Gamma \}
\text{the immediately lower accompanying 1}
\text{following them externally to the chart involved, in counterclockwise fashion, (like}
i \vec{f}(\ldots)}
\text{expression } \vec{f}(\ldots, \ldots)\text{ containing, between the first pair of parentheses, (}
\ldots, \ldots)\text{, the symbols of the oriented 7-cycles containing } \vec{f} \text{ and } (\vec{e})^{-1} \text{ in the other two charts in each case, where } (\vec{f})^{-1} \text{ and } \vec{e} \text{ are the corresponding arcs in}
i \vec{f}, \text{ and containing, between the second pair of parenthesis, } (\ldots), \text{ the symbol following them externally to the chart involved, in counterclockwise fashion, (like}
\text{the immediately lower accompanying } 1^1(3^36^3)(5^2) \ldots).\]

This codifies the assembly of the three charts into the claimed graph \( \Gamma' \). Moreover, the 24 oriented 7-cycles \( i^j \) can be filled each with a corresponding 2-cell, so that because of the cancelations of the two opposite arcs on each edge of \( \Gamma' \) (for having opposite orientations makes them mutually cancelable), \( \Gamma' \) becomes embedded into a closed orientable surface \( T_3 \). As for the genus of \( T_3 \), observe that
\[ |V(\Gamma')| = 2 \times 28 = 56 \text{ and } |E(\Gamma')| = 2|E(\Gamma)| = 2 \times 42 = 84, \]
so that by the Euler characteristic formula for \( T_3 \) here,
\[ |V(\Gamma')| - |E(\Gamma')| + |F(\Gamma')| = 56 - 84 + 24 = -4 = 2 - 2g(T_3), \]
and thus \( g = 3 \), so \( T_3 \) is a 3-torus. This yields the Klein map of Coxeter notation \{7, 3\}_8. (See [21, 18, 17] and note that the Petrie polygons of this map are 8-cycles).

**Theorem 3** The Klein graph \( \Gamma' \) is both a \( \{C_7\}_{P_2} - \text{UH} \) graph and a \( \{C_7\}_{\vec{P}_2} - \text{UH} \) digraph, composed by 24 (oriented) 7-cycles that yield the Klein map \{7, 3\}_8 in \( T_3 \).

For the Klein map \{7, 3\}_8, the 3-torus appeared originally dressed as the Klein quartic \( x^3y + y^3z + z^3x = 0 \), a Riemann surface and the most symmetrical curve of genus 3 over the complex numbers. The automorphism group for this Klein map is \( PSL(2, 7) = GL(3, 2), ([4]) \), the same automorphism group of \( F \), whose index is 2 in the common automorphism groups of \( \Gamma \), \( \Gamma' \) and \( \Gamma'' \).
**Corollary 4** The Klein quartic graph $\mathcal{K}$, whose vertices are the 7-cycles $i^j$ of $\Gamma'$, with adjacency between two vertices if their representative 7-cycles have a pair of O-O arcs, is regular of degree 7, chromatic number 8 and has a natural triangular $T_3$-embedding yielding the dual Klein map $\{3,7\}_8$.

**Proof.** Each vertex $i^j$ of $\mathcal{K}$ is assigned color $i \in \{0\} \cup J_7$. Also, we have a partition of $T_3$ into 24 connected regions, each region having exactly seven neighboring regions, with eight colors needed for a proper map coloring. \hfill \Box

4 Final remarks

Following the remarks made after Theorems 1 and 2, it can be said that the zipping method of Section 3 can be adapted to other graphical situations; to begin with, the Pappus graph, the Desargues graph and the Biggs-Smith graph, the last one yielding the Menger graph of a self-dual (102,4)-configuration, what may be called a $\{K_4, L(Q_3)\}_{K_2}$-UH graph, in a similar way in which the graph of [9] is a $\{K_4, K_{2,2,2}\}_{K_2}$-UH graph, where $L(Q_3)$ is the line graph of the 3-cube graph $Q_3$. More specifically, the Biggs-Smith graph yields, by means of an adequate zipping procedure, a connected 12-regular graph which is the union of 102 copies of $L(Q_3)$ without common squares as well as the edge-disjoint union of 102 copies of $K_4$, with each triangle (edge) as the intersection of exactly two (four) copies of $L(Q_3)$. Also, generalizing on zipping results over the Desargues graph, it can concluded that the line graph $L(K_n)$, with $n \geq 4$, is a tightly fastened $\{K_{n-1}, K_3\}_{K_2}$-UH graph with $n$ copies of $K_{n-1}$ and $\binom{n}{3}$ copies of $K_3$.

A final remark is that the role played by the Heawood graph $\Gamma''$ in the construction of the so-called Ljubljana semi-symmetric graph [3, 7], which is an 8-cover of $\Gamma''$, makes us wonder whether there are any more relations between this 8-cover and both $\Gamma$ and $\Gamma'$, derived all ultimately from $\Gamma''$.

References


