For any positive integer \( n \), the determination of all the 2-covering graphs \( M_n \) of the complete graph \( K_n \) on \( n \) vertices is given, among which there is only one that is connected and with maximal automorphism group. This graph happens to be bipartite. It is shown that an \( i \)-covering graph of \( K_n \) is hamiltonian for \( i < 4 \). Properly minimal connected nonhamiltonian covering graphs of \( K_n \) are obtained, which are 4-coverings of \( K_n \). Also, nonhamiltonian connected 6-coverings of \( K_n \) are constructed.

1. INTRODUCTION

All graphs considered are finite without loops and multiple edges. Let \( H=(V(H),E(H)) \) be a graph. If \( v \in V(H) \) let \( d(v) \) = degree of \( v \) in \( H \) and let the \textit{star subgraph} \( H^*(v) \) associated to \( v \) be defined as the unique subgraph of \( H \) isomorphic to \( K_{1,d(v)} \) in which \( v \) has degree \( d(v) \). Recall from [4] that if \( H \) and \( G \) are graphs then a full homomorphism \( f:H \rightarrow G \) is a graph homomorphism from \( H \) onto \( G \) inducing surjective vertex and edge correspondences \( f_V:V(H) \rightarrow V(G) \) and \( f_E:E(H) \rightarrow E(G) \). Recall that a full homomorphism is said to be a \textit{covering (graph) map} if it restricts to an isomorphism on each star subgraph graph of \( H \). If \( f:G' \rightarrow G \) is a covering map we say that \( G' \) is the \textit{covering graph} of \( G \) associated to \( f \). Other definitions of a covering graph are given in [1, pg. 127], [6] and [9, pg. 160], the latter in terms of voltages.

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It can be seen that all these definitions are equivalent.

THEOREM 1. If $G'$ and $G$ are connected graphs and $f: G' \rightarrow G$ is a covering map then $f$ has $|f^{-1}(v)|$ independent of $v \in V(G)$, so that $|V(G')|/|V(G)| \in \mathbb{Z}$.

PROOF. Let $v \in V(G)$. Say that $f^{-1}(v) = \{w_1, \ldots, w_k\}$. If $vv' \in E(G)$ then $|f^{-1}(v')| = k$ for there are exactly $k$ disjoint edges $w_1w'_1, \ldots, w_kw'_k$ projecting onto $vv'$ via the covering map $f$. Let $v' \in V(G) \setminus \{v\}$. Since $G$ is connected, then there is a path $(v = v_1, v_2, \ldots, v_r = v')$. By induction on $i = 1, \ldots, r$ it is seen that $|f^{-1}(v_i)| = k$.

Under the hypotheses of Theorem 1, we denote $\mu(G') = |V(G')|/|V(G)|$, and say that $G'$ is a $\mu(G')$-covering of $G$.

A covering graph $G'$ of $G$ is said to be properly minimal if $\mu(G')$ is minimal among all $\mu(H)$, where $H$ runs over all covering graphs of $G$. For any positive integer $n$, the determination of all the properly minimal covering graphs of the complete graph $K_n$ on $n$ vertices is given below. These are 2-coverings of $K_n$. Furthermore, it is shown that every $i$-covering graph of $K_n$ is hamiltonian, for $i \leq 4$. Properly minimal nonhamiltonian covering graphs of $K_n$ are determined for every integer $n \geq 3$. These are 4-coverings of $K_n$. Also a 6-covering of $K_n$ consisting of two 3n-cycles covering a Hamilton cycle in $K_n$ and $n-3$ 1-factors is provided for every $n \geq 3$.

2. EXISTENCE OF PROPER MINIMAL COVERING GRAPHS

THEOREM 2. For any graph $G$, there exists a properly minimal covering graph $M(G)$ of $G$ which is a 2-covering of $G$ and a bipartite graph having $2|V(G)|$ vertices and $2|E(G)|$ edges.

PROOF. Assume $V(G) = \{1, \ldots, n\}$. Let $M(G)$ be the graph having $V(M(G)) = \{1, \ldots, n, 1', \ldots, n'\}$ and $E(M(G)) = \{(i, i') : i, i' \in E(G)\}$. The corresponding covering map $m(G): M(G) \rightarrow G$ is given by
m(G)(i)=m(G)(i')=i, for every i ∈ V(G). From μ₀(M(G))=2 we
see that M(G) is properly minimal. The vertex classes of this
bipartite graph are (1,...,n) and (1',...,n').

REMARK 3. The 2-covering graph M(G) of G was defined in [1,
pg. 131] and further studied in [8]. It can be used in the
generation of large bipartite graphs with fixed degree and
diameter, as done in [2].

A graph G is said to be a (2,n)-partite graph if G is
simultaneously a bipartite graph with vertex classes V₁
and V₂ and an n-partite graph with vertex classes W₁,...,Wₙ
such that |Vᵢ ∩ Wⱼ|=1 (i=1,2 ; j=1,...,n). We may say that
M(Kₙ) is the complete (2,n)-partite graph Kₙ,n;2,...,2.
Note in particular that M(Kₙ) is an antipodal covering graph
in the sense of A.Gardiner [5].

From now on assume that V(Kₙ) = [1,n] = {i ∈ Z : 1≤i≤n}.
Given (i,j) ⊆ [1,n], let θᵢ,j be the operation that trans-
forms M(Kₙ) into another graph M(Kₙ;i,j) by replacing the
edges ij' and i'j by new edges ij and i'j'. See Figure 1.
This kind of operation may be performed once more on
M(Kₙ;i,j) and so on repeateadly for the generation of a graph
M(Kₙ;i₁,j₁;i₂,j₂;...;iₖ,jₖ) obtained from M(Kₙ) by the
successive application of θ(i₁,j₁),θ(i₂,j₂),...;θ(iₖ,jₖ),
where 1≤k ∈ Z and ((i₁,j₁),(i₂,j₂),...,(iₖ,jₖ)) is a subset
of the set P[1,n] of subsets of [1,n]. This modification can
be extended adequately for any graph G.

![Figure 1](image-url)
THEOREM 4. Let $M_n$ be a properly minimal covering graph of $K_{2n}$. Without loss of generality we may take $V(M_n)=V(M(K_{2n}))$. Then $M_n=M(K_n;i_1,j_1;i_2,j_2;\ldots;i_k,j_k)$, for some $1<k \in \mathbb{Z}$ and $\{(i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)\} \subseteq P[1,n]$.

PROOF. $M_n$ has $2n$ vertices, since $M(K_{2n})$ is already a properly minimal covering graph of $K_{2n}$ and has $2n$ vertices. A covering map exists from $M_n$ onto $K_{2n}$, so that $M_n$ has two vertices projecting onto the vertex $i \in V(K_{2n})$, for every $i=1,\ldots,n$. Denote these two vertices with $i$ and $i'$, in any order, for every $i=1,\ldots,n$. If $M_n$ is different from $M(K_{2n})$, then there is $(i_1,j_1;i_2,j_2;\ldots;i_k,j_k) \in E(K_{2n})$, such that $i_s$j_s and $i_s'$j_s' are edges of $M_n$ instead of $i_1$j_1 and $i_1'$j_1', for each $s (1\leq s \leq k)$, so that $M_n=M(K_n;i_1,j_1;i_2,j_2;\ldots;i_k,j_k)$.

3. SYMMETRY AND HAMILTONICITY

THEOREM 5. The covering map $m(K_n):M(K_n)\rightarrow K_n$ induces at the automorphism group level the group homomorphism $\text{Aut}(m(K_n)):S_n \times \mathbb{Z}_2 \rightarrow S_n$, given by the projection onto the first component $S_n$, the symmetric group on $n$ elements.

PROOF. A generic element of $\text{Aut}(M(K_n))$ determines a permutation $\sigma$ of $V(M(K_n))$ such that, for every $i \in V(K_n)$, $\sigma(i)$ equals some $j$ or $j'$ (1\leq j \leq 2n; j \neq i) and $\sigma(i)=j$ (j') if $\sigma(i')=j'$ (j). This way, if we denote $a'=b$ iff $b=a'$, where $a,b,a',b' \in V(M(K_n))$, we may write $\sigma=$

\[
\begin{pmatrix}
1 & 2 & \ldots & n & 1' & 2' & \ldots & n'\\
\end{pmatrix}
\begin{pmatrix}
a_1 & a_2 & \ldots & a_n & a'_1 & a'_2 & \ldots & a'_n
\end{pmatrix}
\]

The image of such a permutation $\sigma$ through $\text{Aut}(m(K_n))$ is a permutation of (1,\ldots,n):

\[
\begin{pmatrix}
1 & 2 & \ldots & n \\
a_1 & a_2 & \ldots & a_n
\end{pmatrix}
\] or

\[
\begin{pmatrix}
1 & 2 & \ldots & n \\
a'_1 & a'_2 & \ldots & a'_n
\end{pmatrix}
\]

according to whether $a_1$ equals some $i$ or some $i'$, which establishes the contention.

COROLLARY 6. $M(K_n)$ is the unique properly minimal connected covering graph of $K_{2n}$ having maximal automorphism group.
PROOF. The presence of both edges of the forms $ij$ and $ij'$ in a covering graph $M_n$ of $K_n$, where $i, j \in V(K_n)$, forces the automorphism group of $M_n$ to be smaller than $S_n \times Z_2$. (See Theorem 5). Consequently, only $M(K_n)$ and the graph $M(K_n; 1, 2; \ldots; n-1, n)$ resulting from $M(K_n)$ by replacing each pair $(ij, i'j')$ of edges by another pair $(ij, i'j')$ of edges have automorphism group $S_n \times Z_2$. From these, only $M(K_n)$ is connected. Note that $S_n \times Z_2$ which is thus the maximal automorphism group happening in the found family of properly minimal covering graphs of $K_n$.

THEOREM 7. If $M_n$ is a properly minimal connected covering graph of $K_n$ then $M_n$ is hamiltonian.

PROOF. B. Jackson [7] showed that every 2-connected $k$-regular graph on at most $3k$ vertices is hamiltonian. It is easy to see that $M_n$ is 2-connected. Taking $k = n - 1$, note that $M_n = M_{k+1}$ is $k$-regular and it has $2k+2$ vertices. Also, note that $2k+2 \leq 3k$ if $k \geq 2$.

In order to provide in the next section properly minimal connected nonhamiltonian covering graphs of $K_n$, we need the following result.

THEOREM 8. If $M_n$ is a connected 3-covering graph of $K_n$ then $M_n$ is hamiltonian.

PROOF. Yongjin, Zhenhong and Zhengguang [10, 11, 12] had extended the Jackson's result used in Theorem 7 by proving that if $k \geq 6$ then every 2-connected $k$-regular graph on at most $3k+3$ vertices is hamiltonian. In particular, if $M_n$ is as in the hypothesis, then it is easy to see that $M_n$ is 2-connected and has at most $3k+3$ vertices, where $k = n - 1$. The Theorem follows for $n \geq 7$. The rest can be checked by hand or as a part of an alternative proof of the Theorem in [3].
We proceed to the construction of nonhamiltonian connected 4-covering graphs $N_n$ of $K_{n^2}$, for each positive integer $n \geq 3$. Assume that the set of vertices of $K_{n^2}$ coincides with the set $Z_n = \{1, 2, \ldots, n\}$ subjacent to the cyclic group of order $n$. $N_n$ will be constructed adding extra edges to the disjoint union of an $n$-cycle $A = (0_1, 0_2, \ldots, 0_n, 0_1)$ and a $3n$-cycle $B = (1_1, 2_1, 3_1, \ldots, 3_n, 1_1, 2_1, 3_1, \ldots, 3_n, 1_1)$, where $0_1, \ldots, 3_n$ are to be the vertices of $N_n$ and where vertex $i_j$ ($i \in [0, 3], j \in Z_n$) is to project to vertex $j$ of $K_{n^2}$.

Let $m = [1 + n/2]$. The extra edges are:

(i) $0_1^m, 1_2^m, 2_0^m$ and $3_3^m$;

(ii) $i_j k$ with ends $i_j$ and $i_k$ not already used in $A$, $B$ or (i), where $i \in [0, 3]$ and $(j, k) \subseteq Z_n$.

It can be easily seen that $N_n$ is a nonhamiltonian 4-covering graph of $K_{n^2}$, for each integer $n \geq 3$. Figure 2 shows how $N_4$ looks like, where the vertices are labelled according to the notation used for their image vertices in $K_4$.

The construction of a 6-covering graph $N_n$ of $K_{n^2}$ such that the inverse image of the cycle $(1, \ldots, n, 1)$ is the disjoint union of two $3n$-cycles will be now produced for any integer $n \geq 3$. Denote for convenience $V(K_{n^2}) = (1, 2, \ldots, n-1, 0)$. The graph $N_n$ will contain two disjoint cycles $A$ and $B$. The vertices of $K_{n^2}$ will constitute labels for the vertices of $N_n$ so that a sequence of distinct pairwise adjacent vertices...
covering \( V(A) \) \( (V(B) \) has an associated label sequence \((11,12,\ldots,1(n-1),10,21,22,\ldots,2(n-1),20,31,32,\ldots,3(n-1),30)\). These labels constitute a vertex coloring, where each color paints exactly two vertices of \( A \) \( (B) \). In fact \( V(A) \cup V(B) = V(N_n) \). All the remaining edges of \( N_n \) form a bipartite graph \( G_n \) with vertex classes \( V(A) \) and \( V(B) \). Let \( \phi(ij) \) indicate the set of colors of all adjacent vertices to a vertex \( v \) colored \( ij \) in \( G_n \). This does not depend on the cycle \( A \) or \( B \) to which \( v \) belongs. Each \( \phi(ij) \) can be settled in the form of a column vector of colors in such a way that

\[
\Phi_n = (\phi(11),\phi(12),\ldots,\phi(1(n-1)),\phi(10)) \text{ is a color matrix}
\]

\[
\begin{array}{cccccccccccccccccccccccc}
11 & 12 & 13 & 14 & 10 & 15 & 26 & 17 & 28 & 10 & 11 & 22 & 13 & 34 \\
11 & 12 & 13 & 10 & 21 & 22 & 23 & 20 & 31 & 32 & 33 & 30
\end{array}
\]

**FIGURE 3**

![Figure 3](image)

**FIGURE 4**

![Figure 4](image)
with the following property: If the first coordinate \(i\) of a color \(ij\) is considered as an element of the cyclic group \(\mathbb{Z}_2\), so that addition may be performed, then we may denote 
\[
\Phi_n^k = (\Phi(k1), \Phi(k2), \ldots, \Phi(k(n-1)), \Phi(k0))
\]
as the result of uniformly adding \(k-1\) to the first coordinates of the colors of \(\Phi_n^1\). The matrix \((\Phi_n^1, \Phi_n^2, \Phi_n^3)\) can always be set so that \(N_n\) is nonhamiltonian. A pattern to follow in order to obtain this effect is shown in Figure 3: The matrices \(\Phi_n^1\) for \(n=3, \ldots, 11\) are shown in each case from the second to the last rows, where the top rows indicate the labels \(i\) of the corresponding columns \(\Phi(ij)\) displayed immediately below on the corresponding vertical. The rules for the selection of these matrices are as follows:

1. The transpose \(\Phi(11)^t\) of \(\Phi(11)\) is \((13, \ldots, 1(n-1))\);
2. The \(j\)th row of \(\Phi_n^1\) has associated row of second coordinates: \((j+2, j+3, \ldots, j+n, j, j+1)\), where addition is taken modulo \(n\);
3. From the first to the \((n/2-2)\)th row (where \((a) = 1 + [a]\)) it holds that the associated row of first coordinates of colors splits into two subrows, where the second subrow starts at the unique color whose second coordinate is 1; respectively: first subrow: \((1, 2, \ldots, 1, 2, \ldots)\); second subrow: either \((1, 2, \ldots, 1, 2, \ldots, 1, 2, 1)\) or \((1, 2, \ldots, 1, 2, \ldots, 2, 1, 3)\).
4. If \(n\) is even then the first half of the \(1+(n/2-2)\)th row has its associated row of first coordinates of colors as follows: \((1, 2, \ldots, 1, 2, \ldots)\).

Figure 4 shows how \(M\) looks like. By the selection of the matrices \(\Phi_n^1\) it is seen that there are no quadrangles \(Q \subseteq G_n\) with a pair of opposite edges \(e\) and \(e'\) in \(G_n\). Thus \(M_n\) is nonhamiltonian for every \(n > 3\).

[3] provides proofs of Theorem 7 and 8 depending only on the notion of a covering graph, as well as other related hamiltonicity questions. An alternative proof of Theorem 7 in terms of a \(\mathbb{Z}_2\)-voltage assignment in \(E(K_n)\) that determines \(M_n\), uses in [3] a characterization of those 2-coverings of \(K_n\) which are connected, namely: \(M_n\) is disconnected iff the subgraph \(K_n^0\) induced on \(K_n\) by the 0-edges is either
$K_n$ or $K_{n-1}$ or a spanning disjoint union of two cliques.

REFERENCES

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