UNFOLDING m-GENUS TORI AROUND CAYLEY GRAPHS

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ABSTRACT

An interactive development between Topology and Combinatorics is constituted by the embedding of (Cayley) graphs in manifolds. See for example [5], [6], [7] and [9]. A variation of this is the unfolding of surfaces around graphs, specially infinite graphs having a large amount of symmetry, and in particular Cayley graphs of elementary group rings. It is known that an unfolding of the usual l-genus torus $T_l$, which is in fact an abelian Lie group $S^1 \times S^1$, can be unfolded around the Cayley graph of the group ring $\mathbb{Z}[\mathbb{Z}]$ of linear combinations of sixth roots of unity in the complex plane $\mathbb{C}$, (which is nothing but a tesselation of contiguous equilateral triangles in the plane), by lifting to the Lie algebra $\mathbb{R}^1 \times \mathbb{R}^1$ via the exponential map of that Lie group. In this case, it was convenient to identify $T_l$ with the quotient space obtained from a double equilateral triangle contained in the tesselation above by identifying isometrically and with the same orientation opposite sides of the resulting parallelogram. We generalize this to the case of 2n-th roots of unity, for odd $n > 1$, leading to periodical surfaces in Euclidean space $\mathbb{R}^{n-1}$. This generalization is the natural setting to extend the results of [10] and [11] to maximizing the order of graphs of genus greater than 1, for given values of their diameter and degree.

The Graph Theory notation that we use is as in [2]. A graph map $\gamma$ from a simple graph $G' = (V_G', E_G')$ into a graph $(V_G, E_G)$ is a pair of maps $\gamma_V: V_G' \rightarrow V_G$ and $\gamma_E: E_G' \rightarrow E_G$ such that the edge $\gamma_E((v,w))$ is incident to vertices $\gamma_V(v)$ and $\gamma_V(w)$, where $(v,w) \in E_G'$, is incident to vertices $v$ and $w$ in $G'$. A graph map $\gamma = (\gamma_V, \gamma_E)$ is injective, respectively surjective, if both $\gamma_V$ and $\gamma_E$ are injective, resp. surjective.

Let $G$ and $G'$ be graphs embedded respectively in connected surfaces $M$ and $M'$ by means of graph embeddings $f: G \rightarrow M$ and $f': G' \rightarrow M'$, [9], and let $f_G: G \rightarrow G$ be a surjective graph map. The surface $M'$ is an unfolding of $M$ around $G'$ by means of $f_G$ if there exists a piecewise linear map $f_M: M' \rightarrow M$ such that:

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1. $f_{M}$ induces $f_{G}$;
2. $f_{G} = f_{M} \circ f'$, where $\circ$ stands for composition.

In this case, we say that $(f_{M}^{*}, f_{G}^{*}) : (M', G') \to (M, G)$ is a folding map.

A local Riemann surface $M'$ is a surface that looks locally like a Riemann surface, [1]. The main object of this paper is to establish some facts about the following.

**Unfolding Problem:**

Let $M$ be a (piecewise linear or differentiable) surface and let $G, G'$ and $G''$ be graphs such that $G''$ induces $G'$. Given a surjective graph map $f_{G} : G' \to G$ and a graph embedding $\xi : G \to M$, we look out for a (piecewise linear, differentiable or locally Riemann) surface $M'$ containing $G'$ by means of a graph embedding $f' : G' \to M'$, and for a folding map $(f_{M}', f_{G}') : (M', G') \to (M, G)$. If these objects exist, we say that $M'$ is an unfolding of $M$ around $G'$. Moreover, if the embedding $f'$ is the restriction of an embedding $f'' : G'' \to M'$, we still say that $f''$ is an unfolding of $M$ around $G''$.

In fact, we concentrate here into the cases for which $M$ is the orientable surface of genus $m$ or $m$-genus torus $T_{m}$ obtained by attaching a 2-cell to a graph $G$ with one vertex and $m+1$ loops. We want to unfold $T_{m}$ around (an induced subgraph $G'$ of) the graph $G'' = G_{m}$ that we define subsequently.

Let $n = 2m + 1$ and let $V_{m}$ be the group ring $\mathbb{Z}[[Z_{2n}]]$ of linear combinations of unit roots of $2n$ in the complex plane $\mathbb{C}$ with integer coefficients together with the usual complex sum and product. It is easy to see that $V_{m}$ is:

i. A set of countable cardinality.

ii. Dense in the usual topology of $\mathbb{C}$ iff $m > 1$.

iii. Invariant with respect to the natural affine action of the algebra $\mathbb{Z}[[Z_{2n}]]$ on $V_{m}$,

\[ * \, k : \mathbb{Z}[[Z_{2n}]] \times V_{m} \to V_{m}, \]

given by

\[ * \, k(s, t) = s \cdot t, \]

where $s \in \mathbb{Z}[[Z_{2n}]]$, $t \in V_{m}$ and the multiplication in the right hand side is the usual one in the complex plane $\mathbb{C}$.

The Cayley graph, ([9]), of the group $(V_{m}, +)$ can be described as
the graph $G_m$ with set of vertices $V_m$ and adjacency relation given on any pair $v$ and $w$ of elements of $V_m$ by
* $\text{dist}(v,w) = 1$,
where $\text{dist}$ is the usual distance in the complex plane.

Thus, there is a standard geometric representation $K_m = (V_m, E_m)$ of $G_m = (V_m, E_m)$ in $C$ which is:

i. Edge unimodular, i.e. each edge of $K_m$ has unit length, and

ii. Invariant with respect to the natural affine action of $Z[Z_{2n}]$ on $K_m$,

* $\Gamma: Z[Z_{2n}] \times (V_m, E_m) \to (V_m, E_m)$,
given by $\Gamma = (\ell, \partial)$, where $\partial: Z[Z_{2n}] \times E_m \to E_m$ is defined by $\partial(s, (u,v)) = (\ell(u), \ell(v))$.

iii. Countable, that is, both $V_m$ and $E_m$ are countable. In particular, let $\Psi = (v_1, w_1), \ldots, (v_{\beta}, w_{\beta}), \ldots$, be a sequence covering all of $E_m$.

We consider the infinite graph $K_m$ as a cell complex or CW-complex $C_k$, $[8]$, as follows.

1. The 0-dimensional skeleton $C_{k_0}$ is defined to be $V_m$ as a subset, naturally furnished with the discrete topology of $C_m$. For convenience, let $\alpha: V_m \to C_m$ be a (not a continuous) function induced by the identity of $V_m$.

2. The 1-dimensional skeleton $C_{k_1}$ is obtained from $C_{k_0}$ by attaching a 1-cell, (or closed segment), to each pair of $G_k$-adjacent vertices, and this can be performed inductively. In other words, assuming that $\beta$ 1-cells were already attached to $C_{k_0}$, yielding a CW-complex $D_\beta$, let $v$ and $w$ be adjacent vertices in $G_k$ such that $\alpha(v)$ and $\alpha(w)$ are not joined by a path with its interior in $D_\beta - C_{k_0}$ and consider an immersion onto a a 1-cell $e_{\beta+1}$,

* $\psi: \{\alpha(v), \alpha(w)\} \to e_{\beta+1}$

with $\psi((v,w)) = \text{boundary } (e_{\beta+1})$, which is used to obtain the adjunction topological space, $[8])$:

* $D_\beta + 1 = D_\beta \cup e_{\beta+1}$

obtained by the attachment of $e_{\beta+1}$ to $D_\beta$ via $\psi$.

To solve our unfolding problem we take $G'$ to be the induced subgraph
obtained from $G_m$ by deletion of all its horizontal edges, (or alternatively, all edges parallel to a prefixed edge) and notice the following items.

a. It is convenient to have at hand the following characterization of $T_m$

Let $\tau$ be a convex regular $n$-gon and let $L$ be the straight line determined by one of the sides of $\tau$, (or maximal convex subsets of the frontier of $\tau$). A double regular $n$-gon $H$ in the plane is the union of such a $\tau$ and the image of this $\tau$ under a plane reflection whose axis of symmetry is an $L$ as above. Such an $H$ is a $4m$-gon whose boundary is formed by $2m$ pairs of parallel opposite sides $\{s_1, t_1\}, \ldots, \{s_{2m}, t_{2m}\}$, where $s_i$, resp. $t_i$, is a plane segment $A_iB_i$, resp $C_iD_i$, such that $A_iC_i$ and $B_iD_i$ are also parallel, for $i = 1, \ldots, 2m$. Let $f_i : s_i \to t_i$ be an isometric correspondence under the usual metric on the plane, such that $f_i(A_i) = C_i$, for $i = 1, \ldots, 2m$.

The $f_i$ define an equivalence relation $R$ on the boundary of $H$ having the set of vertices of $H$ as one of its equivalence classes and such that the other equivalence classes have cardinality two. It can be easily seen that the quotient topological space $H/R$ coincides with $T_m$.

b. Consider the set $\sigma$ of cycles of length $n$ in $G_m$.

b1. The image of each of these $n$-cycles in $K_m$ is the boundary of a regular $n$-gon having an horizontal side in the usual complex plane representation.

b2. Moreover, $\sigma$ can be partitioned into two classes, namely the sets $\sigma_u$ and $\sigma_r$ of those $n$-cycles whose filling $n$-gons, i.e. convex hulls, have their interiors in the upper and lower semiplanes determined by the extension straight lines of their horizontal sides.

b3. Notice that each horizontal side determined by an $n$-cycle is common to two such cycles, one in $\sigma_u$ and the other one in $\sigma_r$. The sum of these $n$-cycles is a $2m$-cycle whose image in $K_m$ is the boundary of an $H$ as in a. above.

b4. Both $\sigma_u$ and $\sigma_r$ are countable, as is $\sigma$. In fact, let $c_1, \ldots, c_{2s}$ be the subsequence in $\sigma$ of edges in $G_m$ whose images in $K_m$ are horizontal segments, being each $c_i$ the lower (resp. upper) edge of a cycle $g_{u,i}$ or $g_{r,i}$ in $\sigma_u$ (resp. $g_{u,i}$ in $\sigma_r$). Thus $\mu = g_1, \ldots, g_s = g_{u,1}, g_{r,1}, \ldots, g_{u,r}, g_{r,r}$ is a sequence covering $\sigma$.

c. We construct inductively a 2-dimensional CW-complex $M_m$ out of the 1-dimensional CW-complex $K_m$ in the following way.

cl. For each $n$-cycle $g$ in $G_m$, we consider its image $g'$ in $K_m$ and the
convex regular $n$-gon $g'$ formed by $g''$, i.e. the convex hull of $g'$ in the plane. Now consider the union $X$ of $g$ and $K_m$. Strictly speaking, $X$ is the adjunction space $K_m \cup_{\delta} \overline{g}$ of the 2-dimensional cell $g$ to $K_m$ via the embedding $\delta: g'' \to \overline{g}$.

c2. By means of b4. above, we may suppose that cl. was performed for $g=g_1$. Inductively, suppose that we have already obtained the adjunction space $K'_m=K'_m, b$ of $K_m$ by the attachment of the $b$ convex regular $2m$-gons corresponding to the $n$-cycles $g_1, \ldots, g_b$. For each remaining $n$-cycle $g$ in $G_m$, out of $g_1, \ldots, g_b$, consider the image $g'$ of $g$ in $K_m$ and the convex regular $n$-gon $\overline{g}$ formed by $g'$. Consider the union $X$ of $g$ and $K'_m,b'$ which can be considered as the adjunction space

\[ K'_m, b+1 = K'_m, b \cup_{\delta} \overline{g}_{b+1}, \]

where $\delta_{b+1} : \overline{g}_{b+1}$ is the obvious embedding, if $g'_{b+1}$ is the image of $g_{b+1}$ in $K_m,b$ and $\overline{g}_{b+1}$ is its convex hull.

The direct limit $M_\mu$ of the sequence of topological spaces $K'_m, b+1$ generated by this inductive procedure is the disjoint union of all the convex hulls of all the images of elements of the sequence $\mu$ defined in b4 to $K_m$.

c3. Consider an $n$-cycle $\phi$ in $G_m$. In the construction procedure just given, the image of $\phi$ in $K_m$ was filled with a 2-dimensional cell which is geometrically a convex regular $n$-gon $\theta$. $\phi$ contains a horizontal edge $\eta$ (in $C=R^2$) and we may assume that $\theta$ is the upper semiplane to the straight line $\lambda$ determined by $\eta$. Then there is another convex regular $n$-gon $\theta'$ which is obtained from $\theta$ by the reflection with straight line axis $\lambda$. The union $\kappa$ of $\theta$ and $\theta'$ can be considered as a compact subset of $M_m$ and as a double regular $2n$-gon because is isometric (in the local metric inherited from the plane) to the $H$ conceived in a. above. The subgroup of translations of $Z[Z_{2n}]$ acts on $M_m$ taking $\kappa$ to similar isometrical copies in $M_m'$. Let the vertices of $\theta$ be ordered by counterclockwise adjacency around the center of $\theta$:

\[ A_1, A_2, \ldots, A_n, \]

starting with $A_1$ at the leftmost end of $\eta$, with corresponding opposite vertices in $\theta'$ with respect to the center of $\kappa$:

\[ A'_1, A'_2, \ldots, A'_n. \]

Then, $A'_1=A_n$ and $A_1=A'_n$. We define an isometrical identification between each pair of opposite edges $(A_1, A_1+1)$ and $(A'_1, A'_1+1)$, for
i = 1, \ldots, n - 1. Then the quotient topological space obtained from \kappa by the equivalence relation established by the set of these identifications is homeomorphic to the m-genus torus T^m, with m loops incident to a vertex, each loop being the image of a pair of opposite edges with respect to the center of \kappa, and the vertex being the image of the vertices of \kappa. Let \beta: x \mapsto T^m be the quotient map associated to this remark.

Moreover, the images of \kappa through the action of any element of the algebra \( Z[2n] \) are also good to obtain the same \( T^m \), by the corresponding identifications on their boundaries. This means in particular that all the nonhorizontal edges of \( K \) in a particular class of parallelism in the plane are mapped into a particular loop in \( T^m \); and every loop is the image of one such particular parallelism class of edges. The interior points of \( \kappa \) (or any image of it by the action of an element of \( Z[2n] \)) are mapped into the interior points of the 2-dimensional cell whose attachment to the above graph of one vertex and m loops produced \( T^m \). In particular, the action of any translation of \( Z[2n] \) in \( \kappa \) commutes with \( \beta \).

To show that \( M^m \) is an unfolding of \( T^m \) around \( G^m \), we will prove that \( M^m \) is a locally Riemann differentiable surface, in what follows.

In fact, each element \( v \in V^m \) is a vertex of 2n convex regular n-gons \( G_1, \ldots, G_{2n} \), such that \( G_i \) and \( G_{i+1} \) have a unit side \( s_i \) in common, for \( i = 1, \ldots, 2n \), i.e. \( G_i \cap G_{i+1} = s_i \) and the intersection of all these \( G_i \) or \( s_i \) equals \( v \), for \( i = 1, \ldots, 2n \). These n-gons allow the visualization of a conformal structure defined in a neighborhood \( N(v) \) of \( v \) in \( M^m \). Establishing such a neighborhood \( N(v) \) for each \( v \in V^m \), such that the whole collection of the \( N(v) \) covers \( M^m \), allows to see that \( M^m \) is in fact a local Riemann surface.

Alternatively \( M^m \) can be viewed as a periodical minimal surface [3], embedded in \( \mathbb{R}^{2m} \), in the following fashion. Consider the unit 2m-dimensional cube \( C = [0, 1]^{2m} \). Let \( C(a_1, \ldots, a_{2m}) \) be the (1-dimensional side of \( C \)) given by \( C(a_1, \ldots, a_{2m}) = \{(x_1, \ldots, x_{2m}) ; 0 \leq x_i \leq 1 \text{ and } x_j = a_j \text{ for } j \neq i\} \), where exactly one coordinate \( i \) is selected such that all the \( j = 1, \ldots, 2m \) with \( j \neq i \) have \( a_j \) equal either to 0 or to 1, while \( 0 < a_i < 1 \). Consider the path \( D \) given by the composition or union of the following segments, beginning at the origin in \( \mathbb{R}^{2m} \):

\[
* \quad C(\rho, 0, \ldots, 0), C(1, \rho, 0, \ldots, 0), \ldots, C(1, 1, \ldots, 1, \rho),
\]
and consider the path $D'$ obtained from $D$ by reflection through the center of $C$. The composition or union of $D$ and $D'$ is a circuit of the graph given by the 1-dimensional skeleton of $C$, representing a closed simple curve $S$ in the boundary of $C$. There is a minimal surface $M(0, \ldots, 0)$ inside $C$ whose boundary is $S$. Moreover, by an integer coordinate parallel translation $t(b_1, \ldots, b_{2m})$ given by

$$
(c_1, \ldots, c_{2m}) \rightarrow (c_1 + b_1, \ldots, c_{2m} + b_{2m}),
$$

where $(c_1, \ldots, c_{2m}) \in \mathbb{R}^{2m}$, we can take the cube $C = C(0, \ldots, 0)$ onto another cube $C(b_1, \ldots, b_{2m})$. Thus a parallel minimal surface procedure can be performed in $C(b_1, \ldots, b_{2m})$ to produce a minimal surface $M(b_1, \ldots, b_{2m})$ with boundary $S(b_1, \ldots, b_{2m})$ obtained by the translation $t(b_1, \ldots, b_{2m})$ from $S = S(0, \ldots, 0)$.

Furthermore, the union $\text{Mext}(0, \ldots, 0)$ of those $M(b_1, \ldots, b_{2m})$ in $\mathbb{R}^{2m}$ with $(b_1, \ldots, b_{2m})$ varying in the set

$$
(0, \ldots, 0), (0, \ldots, 0, 1), (0, \ldots, 0, 1, 1), \ldots, (0, 1, \ldots, 1),
$$

$$
(1, \ldots, 1), (1, \ldots, 1, 0), (1, \ldots, 1, 0, 0), \ldots, (1, 0, \ldots, 0),
$$

in $Z^{2m}$ is a $C^\infty$-differentiable minimal surface with boundary, homeomorphic to the subspace of $M_m$ formed by the union of the double regular n-gons $\kappa$, (as in c3), sharing the origin of $C$.

So that $\text{Mext}(0, \ldots, 0)$ can be continually extended to a boundaryless periodical $C^\infty$-differentiable minimal surface $Q_m$, ([3]), in such a way that if $a \in Z^{2m}$ is in $Q_m$, then $Q_m$ contains $\text{Mext}(a)$.

We invest $M_m$ with the differentiable structure provided by $Q_m$, so that we can concentrate our results in the following.

**Theorem 1.** $M_m$ is a canonical unfolding of $T_m$ around the (induced subgraph, $G'$ of the) Cayley graph $G_m = G_m$ (obtained by deletion of all the edges parallel to a prefixed one) with only one singularity $tm$ in $T_m$ iff $m > 1$. Out of $t_m$, the corresponding folding map $S_m : M_m \to T_m$ onto $T_m$ is a topological and differentiable covering. Moreover, the inverse image of $t_m$ through $S_m$ is $V_m$; $V_m$ is dense in $C$ unless $m = 1$, case for which $M_m$ is $C$, the Lie algebra of the toral group $T_1$.

**Proof**

From the previous remarks, let us see that we obtained $M_m$ as an unfolding of $T_m$. In fact, the quotient map

$$
\omega = \omega(0, \ldots, 0) : \kappa = \kappa(0, \ldots, 0) \to T_m
$$

produces, by translations in $\mathbb{Z}[Z_{2n}]$, a collection of maps
\[ \omega(b_1, \ldots, b_{2m}) \cdot \kappa(b_1, \ldots, b_{2m}) \in T_m \]
equivalent to \( \omega \), in the sense that those translations commute with \( \omega \) and each \( \omega(b_1, \ldots, b_{2m}) \) over \( T_m \), and more generally, they commute with pairs of different \( \omega(b_1, \ldots, b_{2m}) \) over \( T_m \). Since the parallelism class defined by an edge of \( \kappa(b_1, \ldots, b_{2m}) \) is mapped onto its corresponding loop in \( T_m \), as a covering out of the distinguished vertex \( t_m \) in \( T_m \), and this happens for each such a parallelism class, we note that a quotient map

\[ f_M : M_m \to T_m \]
is obtained well defined as an extension of each \( \omega(b_1, \ldots, b_{2m}) \). This is further confirmed from \( c3' \), since any element in \( Z[Z_{2m}] \) could define a different family of double regular \( n \)-gons, with horizontal middle edge rotated an angle of \( 2\pi/2n \) radians, (or the equivalent versions in \( Q_m \), which transforms geodesics back and forth into axes), so that the symmetry provided by the \( Z[Z_{2m}] \)-action on \( M_m \) and \( K_m \) allows to homogenize the model given by double regular \( n \)-gons.

We also mentioned the distinguished vertex \( t_m \) in the bouquet of loops from which \( T_m \) is obtained by the attachment of a double regular 2n-gon. If \( m = 1 \), then it is clear that \( t_m \) is not a singular point in \( M_m \), because \( M_m \) is just the plane. However, if \( m > 1 \), we are rotating a total angle of \( (n-2)2\pi \) radians around \( t_m \), say counterclockwise, when attaching convex regular \( n \)-gons side by side, with a common vertex identified to \( t_m \), as part of the induction procedure of \( c1 \) and \( c2' \), implying that \( t_m \) is a singular point of \( M_m \) in this case. Moreover, the inverse image of \( t_m \) in \( M_m \) coincides with \( V_m \), because of our construction. Also, it is clear that \( f_M \) restricted from \( M_m \to V_m \) onto \( T_m - \{ t_m \} \) is a topological and differentiable covering. Finally, \( k = 1 \) leads just \( f_M \) as the standard exponential map of the Lie algebra which is \( M_1 \) over \( T \).

Remark. The unfolding \( M_m \) of \( T_m \) around \( G_m \) or \( K_m \) is canonical in the sense that any other unfolding of \( T_m \) around \( K_m \) is obtained through the action of an element of the algebra \( Z[Z_{2m}] \) on \( M_m \). In the periodical minimal surface version, this means that a parallelism class of geodesics becomes one of straight lines parallel to one of the coordinate axes, and vice versa.

References
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