ASYMPTOTIC DIAMETER OF GRAPHS

Italo J. Dejter
University of Puerto Rico
Rio Piedras, PR 00936-8377
e-mail: italo.dejter@gmail.com

Abstract
A result concerning the asymptotic diameter of graphs is presented.

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The study of the asymptotic diameter of graphs was treated previously for example in [1]. Asymptotic properties of graph families in [2] take to the following conjecture.

Conjecture 1. The asymptotic diameter of a family of graphs $G$ with a common $\Delta(G)$ is a given (radical, logarithmic, . . . ) function of the vertex number of $G$.

The following suggested result contributes to the conjecture in addition to the families presented in [2].

Theorem 2. A radical confirmation of Conjecture 1, for $\Delta = 6$, apart from the graph families $G_0$ and $G_1$ of [2], is given by a collection of families of Cayley graphs $G$ of regular degree $2m \geq 6$ with asymptotic diameter $|V(G)|^{1/m}$, where $3 \leq m \in \mathbb{Z}$.

Proof. (sketch) For $n$ sufficiently large, the undirected Cayley graph $\Lambda_3(n)$ of $\mathbb{Z}_n$ with 0 either adjacent to 1, $n^{1/3}$, $n^{-1/3}$, $n^{2/3}$ and $n^{-2/3}$ or to their nearest integers if $n$ is not a cube, is 6-regular and vertex transitive. If $n$ is not a cube, the following argument is slightly different. Let $x, y \in \mathbb{Z}_n$. There is a path $P_1$ in $\Lambda_3(n)$ from $x$ to $z_1$ with edge differences $\pm n^{2/3}$, where $|y-z_1| \leq n^{2/3}$. The length of $P_1$ is at most $\frac{n}{n^{2/3}} = n^{1/3}$. There is a path $P_2$ in $\Lambda_3(n)$ from $z_1$ to $z_2$, where $|z_2 - z_1| \leq n^{1/3}$, with edge differences $\pm n^{1/3}$. The length of $P_2$ is at most $\frac{n^{2/3}}{n^{1/3}} = n^{1/3}$. There is a path $P_3$ in $\Lambda_3(n)$ from $z_2$ to $y$ with edge differences $\pm 1$. The length of $P_3$ is at most $n^{1/3}$. The concatenation $P = P_1P_2P_3$ from $x$ to $y$ has length at most $3n^{1/3}$. Hence, the diameter of $\Lambda_3(n)$ is at most $3n^{1/3}$. Note that
the length of the shortest path from 0 to \(n/2\) consists entirely of edge differences equal to \(\pm n^{2/3}\), and consequently has \(\frac{n}{2}n^{2/3} = \frac{n^{2/3}}{2}\) edges. Thus, the diameter of \(\Lambda_3(n)\) lies between \(\frac{n^{1/3}}{2}\) and \(3n^{1/3}\).

The argument above can be modified by replacing the denominator 3 in the exponents of \(n\) by any integer \(m > 3\) provided \(n\) is sufficiently large. This leads to a confirmation of Conjecture 1 by means of a family of Cayley graphs \(\Lambda_m(n)\) of \(\mathbb{Z}_n\) with asymptotic diameter \(n^{1/m}\) which is obtained via paths \(P_i\) \((i = 1, \ldots, m)\) of lengths at most \(n^{1/m}\) and edge differences \(\pm n^{(m-i)/m}\) whose orderly concatenation starts at \(x\) and ends at \(y\) with inner concatenation vertices \(z_1, z_2, \ldots, z_{m-1}\) such that \(|y - z_1| \leq n^{(n-1)/n}\) and \(|z_{i+1} - z_i| \leq n^{(n-i)/n}\) for \(1 \leq i \leq m - 2\).

References
