A new approach to the middle levels via a Catalan-number system of numeration

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Abstract

Let \( 0 < k \in \mathbb{Z} \) and let \( M_k \) be the subgraph of the Hasse diagram of the Boolean lattice on \( n = 2k + 1 \) elements induced by its middle levels, namely the \( k \)- and \( (k + 1) \)-levels. A system of numeration according to which the nonnegative integers are written as restricted growth strings, so called by Arndt and Ruskey, has specifically the \( k \)-th Catalan number \( C_k = \binom{n}{k}/n \) expressed as \( 10^k \). This permits a linear ordering of the vertex set of a quotient graph \( R_k \) of \( M_k \) under a dihedral-group action. The Shields-Savage lifting of a Hamilton path in \( R_k \) between certain two distinguished vertices to a Hamilton cycle placed in a dihedrally symmetric fashion in \( M_k \) combines with Kierstead-Trotter lexical matching in \( M_k \) to highlight the relevance of the said system in pursuing the existence, enumeration and sorting (according to that system) of such Hamilton cycles in \( M_k \). We ask whether there are as many such cycles as the \( 2^{2h(n)} \) ones constructed by Mütze in proving the existence of Hamilton cycles (not necessarily placed in dihedrally symmetric fashion) in \( M_k \).

1 Introduction

Let \( n = 2k + 1 \) with \( 0 < k \in \mathbb{Z} \). Assume that the dihedral group \( D_{2n} \) acts on a graph \( G \), and in that case let \( H \) be a \( D_{2n} \)-invariant subgraph of \( G \) [11], pg. 20. If this subgraph \( H \) is a Hamilton cycle of \( G \), then we say that \( H \) is a Hamilton cycle placed in a dihedrally symmetric fashion in \( G \).

In this work each nonnegative integer is expressed as a restricted growth string (or RGS) \( \alpha \), so called by Arndt and Ruskey [2, 17]. This expression is in terms of the Catalan numbers \( C_k = \frac{1}{n} \binom{n}{k} \) (A000108 in [21]), meaning that \( C_k \) becomes \( \alpha = 10^k \), observed by Arndt in [2] pg. 325. A system of numeration (A239903 in [21]) composed by the RGSs \( \alpha \) is presented in Sections 2-3 below, taking in Section 12 to a dihedrally symmetric version of Hávai’s [12] or Buck-Wiedemann’s [4] conjecture proved by Mütze [16] on the existence of Hamilton cycles (not necessarily placed in a dihedrally symmetric fashion) in the middle-levels graphs. These graphs, denoted \( M_k \) (\( 0 < k \in \mathbb{Z} \)), are treated from Section 5 on, after associating the RGSs \( \alpha \)
to corresponding \( n \)-tuples \( F(\alpha) \) (Section 4). In turn, the \( n \)-tuples \( F(\alpha) \) become in Section 10 the vertices of a quotient graph \( R_k \) (Sections 6-8) of \( M_k \) under action of the dihedral group \( D_{2n} \) of \( 2n \) elements.

According to Shields and Savage, Lemma 3 of [18], the existence of Hamilton paths in \( R_k \) (in particular extremal in the \( \alpha \) ordering) between certain distinguished vertices yields Hamilton cycles placed in a dihedrally symmetric fashion in \( M_k \). Moreover, these Shields-Savage paths yield the dihedrally symmetric version of Hável’s conjecture mentioned above and combine with Kierstead-Trotter lexical matchings [13] in \( M_k \) to highlight the relevance of the said system in pursuing the existence, enumeration (perhaps less than \( 2^{2\Theta(n)} \), as cited in the next paragraph) and sorting (according to the system) of such paths.

On the other hand, these tasks remain in spite of the establishment of Hável’s conjecture by Mütze (who reviewed its history in [16], to which bibliography we refer), proving that there are \( 2^{2\Theta(n)} \) such cycles. In our treatment below lexical 1-factorizations in the \( M_k \) s via Kierstead-Trotter lexical matchings [13] get a new meaning from Section 9 on. Ammerlaan and Vassilev [1] showed that any Hamilton cycle in a graph \( M_k \) has the same number of edges along every coordinate direction of the \( n \)-cube \( H_n \) (defined in Section 5 below). This is the case of the Shields-Savage construction of a Hamilton cycle \( \eta_k \) placed in a dihedrally symmetric fashion in \( M_k \) for it uses the cyclic nature mod \( n \) of \( M_k \) in taking a Hamilton path \( \xi_k \) as required and with its vertices in 1-1 correspondence with the first \( C_k \) strings \( \alpha \) and whose dihedral unfolding concatenates with its translates mod \( n \) in order to compose \( \eta_k \) (Section 12 below). In fact, such \( \xi_k \) must visit just once each vertex class of \( M_k \) under an adequate equivalence relation (Section 6). The Shields-Savage construction technique (further employed in [19, 20] with no visible association to any system of numeration, naturally related, or not, to the Catalan numbers) motivates our questioning about existence and enumeration of such Hamilton cycles \( \eta_k \). Since each vertex of \( \xi_k \) corresponds uniquely to a \( k \)-word \( \alpha \) in the system of numeration \( A239903 \), also made precise below, we ask about sorting the \( \xi_k \) s according to this system, which would lead to considering extremal \( \xi_k \) s. Thus, we ask: Are there as many Hamilton cycles placed in a dihedrally symmetric fashion in the \( M_k \) s as Mütze showed by his constructions in the general case in [16]? Moreover, can Mütze’s construction be somehow particularized to Hamilton cycles placed in a dihedrally symmetric fashion in the \( M_k \) s?

In continuation to the present approach to the middle levels, in [8] Section 1 a generalized lexical tree is introduced whose nodes are all the \( k \)-words \((1 \leq k \in \mathbb{Z})\). This tree can be inductively constructed in terms of corresponding \( k \)-co-words. Moreover, in [8] Sections 2-7 the lexically colored adjacency tables in Section 11 below are analyzed in terms of \( k \)-words.

2 Catalan number system of numeration

In the proposed system of numeration, the sequence of increasing non-negative integers is represented as a sequence of RGSs

\[
0, 1, 10, 11, 12, 100, 101, 110, 111, 112, 120, 121, 122, 123, 1000, \ldots
\]

(1)

where the subsequence 1, 10, 100, 1000, \ldots, \( 10 \cdots 0 \) (this with \( t \) zeros, represented symbolically by \( 10^t \), where \( t \geq 0 \)), \ldots, corresponds to the numbers \( C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, \ldots \).
\[ \ldots, C_{t+1} = \ldots, \text{etc. Because of this, we refer to this system as the Catalan-number system of numeration.}\] 

The system can be defined formally as follows. The RGSs in (1) represent the consecutive integers from 0 to 14. They can be also written as expressions \( a_{k-1}a_{k-2} \cdots a_{2}a_{1} \) by prefixing 0s in (1) if necessary, for any adequate \( k \). Such expressions \( a_{k-1}a_{k-2} \cdots a_{2}a_{1} \), called \( k \)-words, are defined for \( 1 < k \in \mathbb{Z} \) by means of the following two rules:

1. The leftmost position in a \( k \)-word \( a_{k-1}a_{k-2} \cdots a_{2}a_{1} \), namely at position \( k - 1 \), contains a digit \( a_{k-1} \in \{0, 1\} \).

2. Given a position \( i > 1 \) with \( i < k \) in a \( k \)-word \( a_{k-1}a_{k-2} \cdots a_{2}a_{1} \), then to the immediate right of the corresponding digit \( a_{i} \), the digit \( a_{i-1} \) (meaning at position \( i - 1 \)) satisfies \( 0 \leq a_{i-1} \leq a_{i} + 1 \).

The reader may compare this with the essentially similar Catalan RGSs in Section 15.2 of [2], as well as with the mixed radix systems of numeration [5], including the factorial number, or factoradic, system [9], [10], [14] pg. 192, [15] pg. 12, or A007623 in [21].

Every \( k \)-word \( a_{k-1}a_{k-2} \cdots a_{2}a_{1} \) yields a \((k + 1)\)-word \( a_{k}a_{k-1}a_{k-2} \cdots a_{2}a_{1} = 0a_{k-1}a_{k-2} \cdots a_{2}a_{1} \). A \( k \)-word \( \neq 0 \) stripped of the null digits to the left of the leftmost position that contains digit 1 is called a Catalan string. We also consider Catalan string 0 for the null \( k \)-words, \( 0 < k \in \mathbb{Z} \). The \( k \)-words are ordered as follows: Given any two \( k \)-words, say \( \alpha = a_{k-1} \cdots a_{2}a_{1} \) and \( \beta = b_{k-1} \cdots b_{1}b_{1} \), where \( \alpha \neq \beta \), then \( \alpha \) precedes \( \beta \), written \( \alpha < \beta \), whenever either

(i) \( a_{k-1} < b_{k-1} \) or

(ii) \( a_{j} = b_{j} \), for \( k - 1 \leq j \leq i + 1 \), and \( a_{i} < b_{i} \), for some \( 1 \leq i < k - 1 \).

The order defined on the \( k \)-words this way is said to be their stair-wise order.

**Observation 1.** The sequence of nonzero Catalan strings has the terms corresponding to the Catalan numbers \( C_{1} = 1, C_{2} = 2, C_{3} = 5, C_{4} = 14, \ldots, C_{t+1} = \frac{1}{2t+3} \binom{2t+3}{t+1}, \ldots, \) written respectively as 1, 10, 100, 1000, \ldots, \( 10^{t}, \ldots, \) where \( 0 \leq t \in \mathbb{Z} \). Moreover, there exists exactly \( C_{k+1} \) \( k \)-words \( < 10^{k} \), for each \( k > 0 \).

We also refer to Stanley’s interpretation (u) of Catalan numbers [22], Exercise (u), as mentioned in A239903 of [21].

### 3 Determination of Catalan strings

To determine the Catalan string corresponding to a given decimal integer \( x_{0} \), or vice versa, one may employ Catalan’s triangle \( \mathcal{T} \), that is a triangular arrangement composed by positive integers starting with the following rows \( \mathcal{T}_{j} \), where \( j = 0, \ldots, 8 \):

\[
\begin{array}{cccccccc}
1 \\
1 & 1 \\
1 & 3 & 5 \\
1 & 4 & 9 & 5 \\
1 & 5 & 14 & 28 & 42 \\
1 & 6 & 20 & 48 & 90 & 132 \\
1 & 7 & 27 & 75 & 165 & 297 & 429 & 429 \\
1 & 8 & 35 & 110 & 275 & 572 & 1001 & 1430 & 1430 \\
\end{array}
\]
where a linear reading of the subsequent rows conforms [A009766] in [21]. In fact, the numbers \( \tau_i^j \) in row \( T_j \) of \( T \) \((0 \leq j \in \mathbb{Z})\) satisfy the following properties:

1. \( \tau_0^j = 1 \), for every \( j \geq 0 \);
2. \( \tau_1^j = j \) and \( \tau_j^j = \tau_{j-1}^j \), for every \( j \geq 1 \);
3. \( \tau_i^j = \tau_{i-1}^j + \tau_i^j \), for every \( j \geq 2 \) and \( i = 1, \ldots, j - 2 \);
4. \( \sum_{i=0}^{j} \tau_i^j = \tau_{j+1}^j = C_j \), for every \( j \geq 1 \).

Moreover, a unified formula for the numbers \( \tau_k^j \) \((j = 0, 1, \ldots, k)\) is given by:

\[
\tau_k^j = \frac{(k+j)!(k+j+1)}{j!(k+1)!}
\]

Now, the determination of the Catalan string corresponding to a decimal integer \( x_0 \) proceeds as follows. Let \( y_0 = \tau_k^{k+1} \) be the largest member of the second diagonal of \( T \) with \( y_0 \leq x_0 \). Let \( x_1 = x_0 - y_0 \). If \( x_1 > 0 \), then let \( Y_1 = \{ \tau_k^j \}^{j+b_1}_{j=k} \) be the largest such set of successive terms in the \((k-1)\)-column of \( T \) with \( y_1 = \sum Y_1 \leq x_1 \). Either \( Y_1 = \emptyset \), in which case we take \( b_1 = -1 \), or not, in which case of course \( b_1 = |Y_1| - 1 \). Let \( x_2 = x_1 - y_1 \). If \( x_2 > 0 \), then let \( Y_2 = \{ \tau_k^{j+b_2} \}^{k+b_2}_{j=k} \) be the largest such set of successive terms in the \((k-2)\)-column of \( T \) with \( y_2 = \sum Y_2 \leq x_2 \). Either \( Y_2 = \emptyset \), in which case we take \( b_2 = -1 \), or not, in which case of course \( b_2 = |Y_3| - 1 \). Proceeding this way, we arrive at a null \( x_k \). Then the Catalan string corresponding to \( x_0 \) is \( a_{k-1}a_{k-2} \cdots a_1 \), where \( a_{k-1} = 1, a_{k-2} = 1 + b_1, \ldots, a_1 = 1 + b_k \).

For example, if \( x_0 = 38 \), then \( y_0 = \tau_3^{\tau_4^1} = 14, x_1 = x_0 - y_0 = 38 - 14 = 24, y_1 = \tau_3^{\tau_4^3} = 5 + 9 = 14, x_2 = x_1 - y_1 = 24 - 14 = 10, y_2 = \tau_3^{\tau_4^3} = 5 + 9 = 14, x_3 = x_2 - y_2 = 10 - 9 = 1, y_3 = \tau_3^{\tau_4^1} = 1 \) and \( x_4 = x_3 - y_3 = 1 - 1 = 0 \), so that \( b_1 = 1, b_2 = 2, \) and \( b_3 = 0 \), taking to \( a_4 = 1, a_3 = 1 + b_1 = 2, a_2 = 1 + b_2 = 3 \) and \( a_1 = 1 + b_3 = 1 \), determining the 5-word of 38 to be \( a_4a_3a_2a_1 = 1231 \). If \( x_0 = 20 \), then \( y_0 = \tau_3^{\tau_4^3} = 14, x_1 = x_0 - y_0 = 20 - 14 = 6, y_1 = \tau_3^{\tau_4^3} = 5, x_2 = x_1 - y_1 = 1, y_2 = 0 \) is an empty sum (since its possible summand \( \tau_2^1 > 1 = x_2 \)), \( x_3 = x_2 - y_2 = 1, y_3 = \tau_3^1 = 1 \) and \( x_4 = x_3 - x_2 = 1 - 1 = 0 \), determining the 5-word of 20 to be \( a_4a_3a_2a_1 = 1101 \). Moreover, if \( x_0 = 19 \), then \( y_0 = \tau_3^{\tau_4^3} = 14, x_1 = x_0 - y_0 = 19 - 14 = 5, y_1 = \tau_3^{\tau_4^3} = 5, x_2 = x_1 - y_1 = 5 - 5 = 0 \), determining the 5-word \( a_4a_3a_2a_1 = 1100 \).

Given a Catalan string or \( k \)-word \( a_{k-1} \cdots a_1 \), the considerations above can easily be played backwards to recover the corresponding decimal integer \( x_0 \).

## 4 Descending casting

**Theorem 2.** To each \( k \)-word \( \alpha = a_{k-1} \cdots a_1 \) corresponds an \( n \)-tuple \( F(\alpha) \) whose entries are the integers \( 0, 1, \ldots, k \) together with \( k \) asterisks \( * \) and such that: (A) The leftmost entry of \( F(\alpha) \) is \( k \). (B) Each integer entry to the immediate right of an integer entry \( b \) is an integer less than \( b \). (C) In particular, \( F(0^{k-1}) = k(k - 1)(k - 2) \cdots 210 \ast \cdots \ast \). (D) To each \( k \)-word \( \alpha \neq 00 \cdots 0 \) corresponds a \( k \)-word \( \beta = b_{k-1} \cdots b_1 \) smaller than \( \alpha \) in the stair-wise order of \( k \)-words, differing from \( \alpha \) in exactly one entry, namely \( b_i \neq a_i \) for just one \( i \) with
\(k - 1 \geq i \geq 1\), and maximal under these two restrictions. Moreover, \(F(\alpha) = f_0f_1 \cdots f_{2k}\) is obtained from \(F(\beta) = g_0g_1 \cdots g_{2k}\) by the descending castling operation consisting of:

1. taking \(g_0 = f_0, g_1 = f_1, \ldots, g_{i-1} = f_{i-1}; g_{2k} = f_{2k}, g_{2k-1} = f_{2k-1}, \ldots, g_{2k-i+1} = f_{2k-i+1}\);

2. denoting the \(i\)-substrings formed by the entries in item 1 by \(W^i\) to the left and \(Z^i\) to the right, and writing \(F(\beta) \setminus (W^i \cup Z^i) = X|Y, (X\) concatenated with \(Y)\), where the substring \(Y\) starts at entry \(\ell - 1\), with \(\ell\) being positive and the leftmost entry of \(X\);

3. by noticing that \(F(\beta) = W^i|X|Y|Z^i\), finally taking \(F(\alpha) = W'|Y|X|Z'\).

There is a rooted tree \(T_k\) whose nodes are the \(k\)-words. In fact, \(T_k\) is associated to the stair-wise order of the \(k\)-words. The root of \(T\) is \(0^{k-1}\) and any other \(k\)-word \(\alpha \neq 0^{k-1}\) is a child in \(T_k\) of a \(k\)-word \(\beta\) that differs from \(\alpha\) in just one entry \(a_i\) \((0 < i < k)\). By representing \(T_k\) with the children of every \(k\)-word \(\alpha\) enclosed between parentheses after \(\alpha\) and separating siblings with commas, we can write, say for \(k = 4\): \(T_4 = 000(001,010(011(012)),100(101,110(111(121)),120(121(122(123))))))\).

**Proof.** In sub-item 2 of item (D), we have: \(\textbf{(a)} W^i\) is a proper subsequence of the maximal starting descending integer sequence \(W\) such that \(W \setminus W^i\) starts with \(\ell\), the head of \(X\); \(\textbf{(b)} Z^i\) is solely composed by asterisks. The \(k\)-tuple \(\alpha\) contains precise instructions for the recursive operation of descending castling to work as prescribed in the statement, away from those \(W^i\) and \(Z^i\) equally long in \(F(\beta)\) and \(F(\alpha)\): By respecting the rule that in every sequence of applications of sub-items 1-3 along descending paths in \(T_k\), unit augmentation of \(a_i\) for larger values of \(i\), \((0 < i < k)\), must occur first, and only then in descending order of \(i\), thus thinning the inner sub-string \(X|Y\) after each application, the resulting process effectively preserves the stated properties, for by changing the order of the appearing sub-strings \(X\) and \(Y\), that have their first elements being respectively \(\ell\) and \(\ell - 1\) in successive decreasing order, the descending nature of this operation is effectively guaranteed.

Let us illustrate and exemplify the descending castling operation. For \(k = 2, 3, 4\), the \(k\)-words \(\alpha\) are presented in their stair-wise order in Table I, both on the left and the right columns, and their corresponding images under \(F\), or \(F\)-images, on the penultimate column. In each of the three listings, each with \(C_k\) rows \((C_2 = 2, C_3 = 5\) and \(C_4 = 14)\), the columns are filled, from the second row on, as follows: \(\textbf{(a)} \alpha\), appearing in downward stair-wise order; \(\textbf{(b)} \beta\) as in item \(\textbf{(D)}\) above, which allows to determine the subindex \(i\) in item \(\textbf{(d)}\) below, where \(\alpha\) and \(\beta\) differ; \(\textbf{(c)} F(\beta)\), from the penultimate column in the previous row; \(\textbf{(d)}\) the only subindex \(i\) \((k - 1 \geq i \geq 1)\) for which the \(i\)-th entries in \(\alpha\) and \(\beta\) differ, with \(\beta\) as large as possible such that \(\beta < \alpha\); \(\textbf{(e)}\) the decomposition \(W^i|Y|X|Z^i\) of \(F(\beta)\); \(\textbf{(e)}\) the result of the descending castling operation; \(\textbf{(f)}\) the corresponding re-concatenation in the column \(F(\alpha)\); and \(\textbf{(g)}\) again \(\alpha\), as \(F^{-1}(F(\alpha))\).

Clearly, for each \(k\)-word \(\alpha\) different from the zero \(k\)-word there exists a well determined \(\beta\) obtained as indicated in item \(\textbf{(D)}\) and observable by comparing the first two columns in Table I, which in addition highlights the sub-index \(i\) of the fourth column. On the other hand, not all the \(n\)-tuples satisfying items \(\textbf{(A)}\)-\(\textbf{(B)}\) above appear in the finite recursion implied by
items (C)-(D). For example, \( \beta' = 431 \ast 2 \ast \ast 0 \ast \) is not image of a permissible \( \alpha \) for \( k = 4 \), (see Table I, for \( k = 4 \)). In fact, trying to apply sub-items (1)-(3) for \( i = 1 \) to \( \beta' \) would result in \( \alpha' = 42 \ast \ast 031 \ast \ast \) which does not respect item (B). For \( i = 2 \), we would have \( \beta' \) decomposing as \( 43|1 \ast 2 \ast \ast |0 \ast \ast \), and since the first sub-word of the middle part starts with 1, we would like to take 0 as the second sub-word, but 0 is outside the middle part. Again this is not a case in our treatment, as indicated in item (b) of the proof of Theorem 2.

### Table I

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( F(\beta) )</th>
<th>( i )</th>
<th>( W^i \mid X \mid Y \mid Z^i )</th>
<th>( W^i \mid Y \mid X \mid Z )</th>
<th>( F(\alpha) )</th>
<th>( \alpha' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>210**</td>
<td>1</td>
<td>2 ( \mid 1 \ast 0 \ast \ast )</td>
<td>2 ( \mid 0 \ast 1 \ast )</td>
<td>210<em>1</em></td>
<td>0</td>
</tr>
<tr>
<td>00</td>
<td>00</td>
<td>3210***</td>
<td>2</td>
<td>32 ( \mid 1 \ast 0 \ast \ast )</td>
<td>30 ( \mid 1 \ast \ast 2 \ast )</td>
<td>310<em>2</em></td>
<td>01</td>
</tr>
<tr>
<td>10</td>
<td>00</td>
<td>3210***</td>
<td>2</td>
<td>32 ( \mid 1 \ast 0 \ast \ast )</td>
<td>32 ( \mid 0 \ast 1 \ast )</td>
<td>320<em>1</em></td>
<td>10</td>
</tr>
<tr>
<td>11</td>
<td>10</td>
<td>320*1**</td>
<td>1</td>
<td>3 ( \mid 20 \ast 1 \ast \ast )</td>
<td>3 ( \mid 1 \ast 2 \ast \ast )</td>
<td>31*20**</td>
<td>11</td>
</tr>
<tr>
<td>12</td>
<td>11</td>
<td>31*20**</td>
<td>1</td>
<td>3 ( \mid 1 \ast 2 \ast 0 \ast \ast )</td>
<td>3 ( \mid 0 \ast 1 \ast 2 \ast )</td>
<td>30<em>1</em>2*</td>
<td>12</td>
</tr>
<tr>
<td>000</td>
<td>000</td>
<td>43210****</td>
<td>1</td>
<td>43 ( \mid 210 \ast \ast \ast )</td>
<td>4210<em>3</em></td>
<td>4210<em>3</em></td>
<td>001</td>
</tr>
<tr>
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<td>000</td>
<td>43210****</td>
<td>2</td>
<td>43 ( \mid 210 \ast \ast \ast )</td>
<td>4310<em>2</em></td>
<td>4310<em>2</em></td>
<td>010</td>
</tr>
<tr>
<td>011</td>
<td>010</td>
<td>4310<strong>2</strong></td>
<td>1</td>
<td>43 ( \mid 210 \ast \ast \ast )</td>
<td>43 ( \mid 310 \ast \ast \ast )</td>
<td>42<em>310</em></td>
<td>011</td>
</tr>
<tr>
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<td>011</td>
<td>42<em>310</em>**</td>
<td>1</td>
<td>42 ( \mid 310 \ast \ast \ast )</td>
<td>42 ( \mid 10 \ast \ast 3 \ast )</td>
<td>410<em>2</em>3*</td>
<td>012</td>
</tr>
<tr>
<td>100</td>
<td>000</td>
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<td>3</td>
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<td>432 ( \mid 0 \ast \ast \ast )</td>
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<td>100</td>
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<td>42 ( \mid 10 \ast \ast 3 \ast )</td>
<td>420<em>1</em>3*</td>
<td>101</td>
</tr>
<tr>
<td>110</td>
<td>100</td>
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<td>43 ( \mid 120 \ast \ast \ast )</td>
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</tr>
<tr>
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<td>110</td>
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<td>43 ( \mid 210 \ast \ast \ast )</td>
<td>42 ( \mid 30 \ast 1 \ast \ast )</td>
<td>420<em>31</em></td>
<td>111</td>
</tr>
<tr>
<td>112</td>
<td>111</td>
<td>420<em>31</em>**</td>
<td>1</td>
<td>42 ( \mid 310 \ast \ast \ast )</td>
<td>41 ( \mid 20 \ast \ast 3 \ast )</td>
<td>4120<em>1</em></td>
<td>112</td>
</tr>
<tr>
<td>120</td>
<td>110</td>
<td>431<em>20</em>**</td>
<td>2</td>
<td>43 ( \mid 210 \ast \ast \ast )</td>
<td>43 ( \mid 0 \ast 1 \ast \ast )</td>
<td>4301<em>2</em></td>
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</tr>
<tr>
<td>121</td>
<td>120</td>
<td>430<em>12</em>**</td>
<td>1</td>
<td>43 ( \mid 0 \ast 1 \ast \ast )</td>
<td>42 ( \mid 30 \ast 1 \ast \ast )</td>
<td>42<em>30</em>1*</td>
<td>121</td>
</tr>
<tr>
<td>122</td>
<td>121</td>
<td>42<em>30</em>**</td>
<td>1</td>
<td>42 ( \mid 30 \ast 1 \ast \ast )</td>
<td>41 ( \mid 2 \ast 3 \ast \ast )</td>
<td>41<em>2</em>30*</td>
<td>122</td>
</tr>
<tr>
<td>123</td>
<td>122</td>
<td>41<em>230</em>**</td>
<td>1</td>
<td>41 ( \mid 2 \ast 3 \ast \ast )</td>
<td>40 ( \mid 1 \ast 2 \ast 3 \ast )</td>
<td>40<em>1</em>2<em>3</em></td>
<td>123</td>
</tr>
</tbody>
</table>

One may perform sub-items 1-3 departing from a \( k \)-word \( \beta \) independently of a specific \( \alpha \) by taking a subindex \( i \) in the text of item (D) and after replacing numbers and asterisks respectively by 0s and 1s, obtaining a \( k \)-work \( \alpha' \) provided already by the textual item (D). Say we depart for \( k = 3 \) from \( F(\beta) = F(01) = 310*2*2* \) and take \( i = 1 \). Sub-items 1-3 leads here to \( F(\alpha') = 30*2*21* \), which yields 0011001, a translation mod 7 of \( 31*20** = F(11) \), obtained already in a different way in Table I.

To each \( F(\alpha) \) corresponds a binary \( n \)-tuple \( \phi(\alpha) \) of weight \( k \) obtained by replacing each integer entry in \( \{0,1,\ldots,k\} \) by 0 and each asterisk * by 1. By attaching the entries of \( F(\alpha) \) as subscripts to the corresponding entries of \( \phi(\alpha) \), a subscripted binary \( n \)-tuple \( \hat{\phi}(\alpha) \) is obtained. Let \( \mathcal{N}(\phi(\alpha)) \) be given by the reverse complement of \( \phi(\alpha) \), that is

\[
\text{if } \phi(\alpha) = a_0 a_1 \cdots a_{2k}, \text{ then } \mathcal{N}(\phi(\alpha)) = \bar{a}_{2k} \cdots \bar{a}_1 \bar{a}_0, \tag{2}
\]
where $\bar{0} = 1$ and $\bar{I} = 0$. A subscripted version $\tilde{\mathbf{R}}$ of $\mathbf{R}$ is immediately obtained for $\tilde{\phi}(\alpha)$. Observe that every image under $\mathbf{R}$ is an $n$-tuple of weight $k + 1$ and has the 1s with integer subscripts and the 0s with asterisk subscripts. The integer subscripts reappear from Section 9 to Section 11 as lexical colors [13]. Table II illustrates the notions just presented, for $k = 2, 3$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\phi(\alpha)$</th>
<th>$\tilde{\phi}(\alpha)$</th>
<th>$\tilde{\mathbf{R}}(\phi(\alpha)) = \mathbf{R}(\tilde{\phi}(\alpha))$</th>
<th>$\mathbf{R}(\phi(\alpha))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>000111</td>
<td>0,0,0,1,0,1,1</td>
<td>0,0,0,1,0,1,2</td>
<td>00111</td>
</tr>
<tr>
<td>1</td>
<td>001011</td>
<td>0,0,0,1,0,1,1</td>
<td>0,0,1,0,1,0,1</td>
<td>01011</td>
</tr>
<tr>
<td>00</td>
<td>0000111</td>
<td>0,0,0,0,1,1,1,1</td>
<td>0,0,0,0,1,1,2,1</td>
<td>0000111</td>
</tr>
<tr>
<td>01</td>
<td>0001101</td>
<td>0,0,0,1,1,0,1</td>
<td>0,0,1,0,1,1,2</td>
<td>0100111</td>
</tr>
<tr>
<td>10</td>
<td>0001011</td>
<td>0,0,0,1,0,1,1</td>
<td>0,0,1,0,1,0,1,2</td>
<td>0010111</td>
</tr>
<tr>
<td>11</td>
<td>0010011</td>
<td>0,0,1,1,0,0,1,1</td>
<td>0,0,1,0,1,2,1,1</td>
<td>0010111</td>
</tr>
<tr>
<td>12</td>
<td>0010101</td>
<td>0,0,1,1,0,1,0,1,1</td>
<td>0,1,0,1,0,1,0,1,3</td>
<td>0010101</td>
</tr>
</tbody>
</table>

An interpretation of this related to the middle-levels graphs is started at the end of Section 6 in relation to the subscripts $0, 1, \ldots, k$ and concluded as Corollary 6 at the end of Section 10.

5 Middle-levels graphs

Let $1 < n \in \mathbb{Z}$. The $n$-cube graph $H_n$ is the Hasse diagram of the Boolean lattice on the coordinate set $[n] = \{0, \ldots, n-1\}$. Vertices of $H_n$ are cited in three different ways:

(a) as the subsets $A = \{a_0, a_1, \ldots, a_{r-1}\} = a_0a_1\cdots a_{r-1}$ of $[n]$ they stand for, where $0 < r \leq n$;

(b) as the characteristic $n$-vectors $B_A = (b_0, b_1, \ldots, b_{n-1})$ over the field $\mathbb{F}_2$ that the subsets $A$ represent, meaning they are given by $b_i = 1$ if and only if $i \in A$ ($i \in [n]$), and represented for short by $B_A = b_0b_1\cdots b_{n-1}$;

(c) as the polynomials $\epsilon_A(x) = b_0 + b_1x + \cdots + b_{n-1}x^{n-1}$ associated with the vectors $B_A$.

A subset $A$ as in (a) is said to be the support of the vector $B_A$ in (b). For each $j \in [n]$, the $j$-level $L_j$ is the vertex subset in $H_n$ formed by those $A \subseteq [n]$ with $|A| = j$. For $1 \leq k \in \mathbb{Z}$, the middle-levels graph $M_k$ is defined as the subgraph of $H_n$ induced by $L_k \cup L_{k+1}$.

6 Quotient graph under cyclic action

By viewing the vertices of $M_k$ as polynomials, as in item (c) of Section 5, an equivalence relation $\pi$ is seen to exists in the vertex set $V(M_k)$ of $M_k$ by means of the logical expression:

$$
\epsilon_A(x)\pi\epsilon_{A'}(x) \iff \exists i \in \mathbb{Z} \text{ such that } \epsilon_{A'}(x) \equiv x^i\epsilon_A(x) \pmod{1 + x^n}.
$$
\[ M_2/\pi = \begin{array}{c|c|c|c|c}
\ell/\pi & (0011) & (0010) & (0111) & (0101) \\
\hline
0 & 1 & 2 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 \\
\end{array} \]

\[ \gamma_2 : \begin{array}{c|c|c|c|c}
\gamma_2 & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{array} = R_2 \]

Figure 1: Reflection symmetry of \( M_2/\pi \) about a line \( \ell/\pi \) and resulting graph map \( \gamma_2 \)

This implies that there exists a quotient graph \( M_k/\pi \) under a regular (i.e. transitive and free) action

\[ \Upsilon' : \mathbb{Z}_n \times M_k \rightarrow M_k \] (3)

given by \( \Upsilon'(i, v) = v(x)x^i \mod 1 + x^n \) in polynomial terms as in item (c) of Section 5, where \( v \in V(M_k) \) and \( i \in \mathbb{Z}_n \). Here, \( M_k/\pi \) is the graph whose vertices are the equivalence classes of vertices of \( M_k \) under \( \pi \) and whose edges are the equivalence classes that \( \pi \) induces on the edge set \( E(M_k) \) of \( M_k \).

For example, \( M_2/\pi \) is the domain of the graph map \( \gamma_2 \) suggested in Figure 1 and associated with reflective symmetry of both \( M_2/\pi \) and \( M_2 \) about respective dashed vertical lines \( \ell/\pi \) and \( \ell \) acting as symmetry axes (generalized below in Section 7) with \( V(M_2/\pi) = L_2/\pi \cup L_3/\pi \), where \( L_2/\pi = \{(00011), (00101)\} \) and \( L_3/\pi = \{(00111), (01011)\} \), each \( \pi \)-class expressed between parentheses about one of its representatives written as in (b) of Section 5 and composed by the following elements (of \( L_2/\pi \) and \( L_3/\pi \)):

\[ L_2/\pi = \{(00011) = (00011, 10001, 11000, 01100, 00110), (00101) = (00101, 10010, 01001, 10100, 01010)\}; \]
\[ L_3/\pi = \{(00111) = (00111, 10011, 11011, 11100, 01101), (01011) = (01011, 10101, 11010, 10101, 01101)\}. \]

Let \( k > 1 \) be a fixed integer. We associate with each binary weight-\( k \) \( n \)-tuple \( F(A) \) the class \( (F(A)) \) generated by \( F(A) \) in \( L_k/\pi \) and the class \( (N(F(A))) \) generated by \( N(F(A)) \) in \( L_{k+1}/\pi \). These associations start the interpretation of Section 4 above to be concluded as Corollary 6 in Section 10.

7 Reflective-symmetry graph involution

A graph involution of a graph \( G \) is a graph map \( \mathcal{R} : G \rightarrow G \) such that \( \mathcal{R}^2 \) is the identity graph map. Clearly, a graph involution is a graph isomorphism. In a way similar to the example for \( k = 2 \) in Section 6, but now for any \( k \geq 2 \), and in order to make explicit a graph involution \( \mathcal{R} \) of \( M_k/\pi \) given by reflective symmetry, as suggested in Figure 1 about the symmetry axis \( \ell/\pi \), we want now to list and represent vertically the vertex parts \( L_k/\pi \) and \( L_{k+1}/\pi \) of \( M_k/\pi \) (resp. \( L_k \) and \( L_{k+1} \) of \( M_k \)) by placing their vertices into pairs, each pair displayed on an horizontal line with its two composing vertices equidistant from a dashed vertical line \( \ell/\pi \) (resp. \( \ell \)), like the \( \ell/\pi \) in the representation of \( M_2/\pi \) in Figure 1. To specify the desired vertex setting, the definition of \( \mathcal{R} \) in display (2) can be immediately extended to a bijection \( \mathcal{R} : L_k \rightarrow L_{k+1} \), where the image of an element of \( L_k \) through \( \mathcal{R} \) is again said to be its reverse complement. Let us take each resulting horizontal vertex pair to be of the form \( (B_A, N(B_A)) \) and ordered from left to right. Let \( \rho_i : L_i \rightarrow L_i/\pi \) be the canonical projection given by assigning \( b_0b_1\cdots b_{n-1} \in L_i \) to \( (b_0b_1\cdots b_{n-1}) \in L_i/\pi \), for \( i = k, k + 1, \ldots \). 

and let \( \mathcal{R}_\pi : L_k / \pi \to L_{k+1} / \pi \) be given by \( \mathcal{R}_\pi((b_0 b_1 \cdots b_{n-1})) = (\bar{b}_{n-1} \cdots \bar{b}_1 \bar{b}_0) \). Then \( \mathcal{R}_\pi \) is a bijection and we have the commutative identity \( \rho_{k+1} \mathcal{R} = \mathcal{R}_\pi \rho_k \). In what follows, we say that a non-horizontal edge of \( M_k / \pi \) is a skew edge.

**Theorem 3.** To each skew edge \( e = (B_A)(B_{A'}) \) of \( M_k / \pi \) corresponds a different skew edge \( \mathcal{R}_\pi((B_A)) \mathcal{R}_\pi^{-1}((B_{A'})) \) obtained from \( e \) by reflection on the line \( \ell / \pi \), which is equidistant from \( (B_A) = \mathcal{R}_\pi^{-1}((B_A')) \in L_k / \pi \) and \( \mathcal{R}_\pi((B_A)) = (B_{A'}) \in L_{k+1} / \pi \). Thus: (i) the skew edges of \( M_k / \pi \) appear in pairs, having their endpoints in each pair forming two pairs of horizontal vertices equidistant from \( \ell / \pi \); (ii) the horizontal edges of \( M_k / \pi \) have multiplicity \( \leq 2 \).

**Proof.** With the representation adopted for the vertices of \( M_k \), the skew edges \( B_A B_{A'} \) and \( \mathcal{R}_\pi^{-1}(B_{A'}) \mathcal{R}(B_A) \) of \( M_k \) are now seen to be reflection of each other about \( \ell \) and also having their pairs \( (B_A, \mathcal{R}(B_A)) \) and \( (\mathcal{R}_\pi^{-1}(B_{A'}), B_{A'}) \) of endpoints lying each on a corresponding horizontal line. Now, \( \rho_k \) and \( \rho_{k+1} \) extend together to a covering graph map \( \rho : M_k \to M_k / \pi \), since the edges accompany the projections correspondingly, as for example for \( k = 2 \), where:

\[
\begin{align*}
\mathcal{R}((0001)) &= (00011, 00110, 01110, 11100, 00110) = (00111) = (01111), \\
\mathcal{R}^{-1}(01011) &= \mathcal{R}^{-1}(01011, 10110, 10110, 10110) = (00101, 10010, 01001, 01001) = (01011),
\end{align*}
\]

showing the order of the elements in the images or preimages under \( \mathcal{N} \) of the classes modulo \( \pi \) as displayed in Figure 1, that is: presented backwards (from right to left), cyclically between braces, and continuing on the right once one reaches a leftmost brace. This backwards behavior holds for any \( k > 2 \), that is:

\[
\begin{align*}
\mathcal{R}((b_0 \cdots b_{2k})) &= \mathcal{R}((b_0 \cdots b_{2k}, b_{2k} \cdots b_{2k-1}, \cdots, b_1 \cdots b_0)) = (b_{2k} \cdots b_0, b_{2k-2} \cdots b_2, \cdots, b_1 \cdots b_0) = (b_{2k} \cdots b_0), \\
\mathcal{R}^{-1}(b_{2k} \cdots b_0) &= \mathcal{R}^{-1}(b_{2k} \cdots b_0, b_{2k-2} \cdots b_2, \cdots, b_1 \cdots b_0) = (b_{2k} \cdots b_0, b_{2k-2} \cdots b_2, \cdots, b_1 \cdots b_0) = (b_{2k} \cdots b_0),
\end{align*}
\]

for any vertices \( (b_0 \cdots b_{2k}) \in L_k / \pi \) and \( (b'_0 \cdots b'_{2k}) \in L_{k+1} / \pi \). This establishes item (i). Now, every edge of \( M_k \) from a vertex in \( \mathcal{R}^{-1}(v) \) to a vertex in \( \mathcal{R}^{-1}(\rho(v)) \), for some \( v \in L_k \), projects onto an horizontal edge of \( M_k \), while all other edges of \( M_k \) project onto corresponding skew edges of \( M_k \). It is easy to see that an horizontal edge of \( M_k / \pi \) has its endpoint in \( L_k / \pi \) represented by a vertex \( \bar{b}_k \cdots \bar{b}_1 \bar{b}_0 \bar{b}_1 \cdots \bar{b}_0 \in L_k \) so there are \( 2^k \) such vertices in \( L_k \) and less than \( 2^k \) corresponding vertices of \( L_k / \pi \); (for example, \( (0^{k+1}1^k) \) and \( (01) \) (compare with the lifting Lemma 3 of [18]) are endpoints of two horizontal edges, each in \( M_k / \pi \)). To prove that this implies item (ii), we have to see that there cannot be more than two representatives \( \bar{b}_k \cdots \bar{b}_1 \bar{b}_0 \bar{b}_1 \cdots \bar{b}_0 \) and \( c_k \cdots c_1 c_0 c_1 \cdots c_k \) of a vertex \( v \in L_k / \pi \), with \( b_0 = c_0 = 0 \). Let \( v = (d_0 \cdots d_{i+1} \cdots d_{j-1} c_0 \cdots d_{2k}) \), with \( b_0 = d_i \), \( c_0 = d_j \) and \( 0 < j - i \leq k \). A substring \( \sigma = d_{i+1} \cdots d_{j-1} \) with \( 0 < j - i \leq k \) is said to be \( (j-i) \)-feasible if \( v \) fulfills (ii) with multiplicity at least 2. Any \( (j - i) \)-feasible substring \( \sigma \) forces in \( L_k / \pi \) only endpoints \( \omega \) incident to two different (parallel) horizontal edges in \( M_k / \pi \) because periodic continuation mod \( n \) of \( d_0 \cdots d_{2k} \) both to the right of \( d_j = c_0 \) with minimal cyclic substring \( d_{j-1} \cdots d_{i+1} 1 d_{i+1} \cdots d_{j-1} = P_f \) and to the left of \( d_i = b_0 \) with minimal cyclic substring \( 0 d_{i+1} \cdots d_{j-1} 1 d_{j-1} \cdots d_{i+1} = P_f \) yields a two-way infinite string that winds up onto \( (d_0 \cdots d_{2k}) \) corresponding to \( \omega \). For example, the initial feasible substrings \( \sigma \), with ‘o’ indicating the positions \( b_0 = 0 \) and \( c_0 = 0 \), are

\[
(0,(001)), (0,(0011)), (1,(010)), (0^2,(000111)), (01,(001011)), (1^2,(011001)), (0^2,(000001111)), (010,(010101101)), (01^2,(01101)), (101,(01010)), (1^3,(0111000)),
\]

where \( n = 3, 5, 5, 7, 7, 5, 9, 11, 5, 7 \). (However, the substrings \( 0^2 1 \) and \( 10^2 \) are non-feasible). If \( \sigma \) is a feasible substring and \( \overline{\sigma} \) is its reverse complement via \( \mathcal{N} \), then the
possible symmetrical substrings about \( o\sigma o = 0\sigma 0 \) in (the notation of) a vertex \( \omega \) of \( L_k/\pi \) are in order of ascending length:

\[
\begin{align*}
0\sigma 0, \\
0\sigma 0\sigma 0, \\
1\sigma 0\sigma 0\sigma 0, \\
\sigma 0\sigma 0\sigma 0\sigma 0, \\
0\sigma 0\sigma 0\sigma 0\sigma 0\sigma 0, \\
\sigma 0\sigma 0\sigma 0\sigma 0\sigma 0\sigma 0.
\end{align*}
\]

etc., where we use again ‘0’ instead of ‘o’ for the entry immediately preceding (following) the shown central copy of \( \sigma \). Due to this, the finite lateral periods of the resulting \( P_\tau \) and \( P_\ell \) do not allow a third horizontal edge up to returning back to \( b_0 \) or \( c_0 \) since no entry \( e_0 = 0 \) of \( (d_0 \cdots d_{2k}) \) other than \( b_0 \) or \( c_0 \) happens such that \( (d_0 \cdots d_{2k}) \) has a third representative \( \bar{e}_k \cdots \bar{e}_0 \cdots e_k \) (besides \( \bar{b}_k \cdots \bar{b}_0 b_1 \cdots b_k \) and \( \bar{c}_k \cdots \bar{c}_0 c_1 \cdots c_k \)). Thus, those two horizontal edges are produced solely from the feasible substrings \( d_{i+1} \cdots d_{j-1} \) characterized above.

To illustrate the ideas in the proof of Theorem 3, let \( 1 < h < n \) in \( \mathbb{Z} \) be such that \( \gcd(h,n) = 1 \) and let the \( h \)-interspersion \( \lambda_h : L_k/\pi \to L_k/\pi \) be given by \( \lambda((a_0 a_1 \cdots a_n)) \to ((a_0 a_{h2} a_{h1} \cdots a_{n-2h} a_{n-h})) \). For each \( h \) with \( 1 < h \leq k \), there exists at least one \( h \)-feasible substring \( \sigma \) and a resulting associated vertex \( \omega \in L_k/\pi \) as in the proof of the theorem. For example, applying \( h \)-interspersion repeatedly by starting at \( \omega = (0^h1^k) \in L_k/\pi \) produces a number of such vertices \( \omega \in L_k/\pi \). If we assume \( h = 2h' \) with \( h' \in \mathbb{Z} \), then an \( h \)-feasible substring \( \sigma \) has the form \( \sigma = a_1 \cdots a_{h'} \cdots a_1 \), so there are at least \( 2^{h'} = 2^h \) such \( h \)-feasible substrings.

8 Quotient graph under dihedral action

Given a graph \( G \) with an involution \( \kappa : G \to G \), a graph folding of \( G \) is a graph \( H \) whose vertices are the pairs \( \{v, \kappa(v)\} \), where \( v \in V(G) \), and whose edges are the pairs \( \{e, \kappa(e)\} \), where \( e \in E(G) \). Here, \( e \) has end-vertices \( v \) and \( \kappa(v) \) if and only if \( e \) yields a loop in \( H \); otherwise, \( e \) yields a link in \( H \) ([3], pg. 3). Let us denote each pair \( ((B_A), \kappa((B_A))) \) of \( M_k/\pi \), horizontally represented in Section 7, via the notation \( [B_A] \), where \( |A| = k \).

A graph folding \( R_k \) of \( M_k/\pi \) is obtained whose vertices are the pairs \( [B_A] \) and having

1. an edge \( [B_A][B_A'] \) per skew-edge pair \( \{(B_A)\kappa((B_A')); (B_A')\kappa((B_A))\} \);
2. a loop at \( [B_A] \) per horizontal edge \( (B_A)\kappa((B_A)) \). By Theorem 3, there may be up to two loops at each vertex of \( R_k \).

Theorem 4. \( R_k \) is a quotient graph of \( M_k \) under an action \( \Upsilon : D_{2n} \times M_k \to M_k \).

Proof. To define \( \Upsilon \), recall that \( D_{2n} \) is the semidirect product \( \mathbb{Z}_n \rtimes \varrho \mathbb{Z}_2 \) via the group homomorphism \( \varrho : \mathbb{Z}_2 \to \text{Aut}(\mathbb{Z}_n) \) given by taking \( \varrho(0) \) as the automorphism assigning \( i \in \mathbb{Z}_n \) to \( (n-i) \in \mathbb{Z}_n \), and \( \varrho(1) \) is the identity. If \( \ast : D_{2n} \times D_{2n} \to D_{2n} \) indicates multiplication and \( i_1, i_2 \in \mathbb{Z}_n \), then \( (i_1, 0) \ast (i_2, j) = (i_1 + i_2, j) \), but \( (i_1, 1) \ast (i_2, j) = (i_1 - i_2, 1 + j) \), for \( j \in \mathbb{Z}_2 \). Now, set \( \Upsilon((i, j), v) = \Upsilon'(i, \kappa^j(v)) \), for \( i \in \mathbb{Z}_n \) and \( j \in \mathbb{Z}_2 \), where \( \Upsilon' \) was defined in display (3) of Section 6 above. It is easy to see that \( \Upsilon \) is a well-defined action of \( D_{2n} \) on \( M_k \). For example by writing \( (i, j) \cdot v = \Upsilon((i, j), v) \) and \( v = a_0 \cdots a_{2k} \), we have \((i, 0) \cdot v = a_{n-i+1} \cdots a_{2k} a_0 \cdots a_{n-i} = a_0 \cdots a_{n-i} \cdots a_{n-i+1} \cdots a_{2k} \).
\(v'\) and \((0, 1) \cdot v' = \bar{a}_{i-1} \cdots \bar{a}_0\bar{a}_{2k} \cdots \bar{a}_i = (n - i, 1) \cdot v = ((0, 1) \ast (i, 0)) \cdot v\), leading to one instance of the compatibility condition \(((i, j) \ast (i', j')) \cdot v = (i, j) \cdot ((i', j') \cdot v)\) that a group action must satisfy, (together with the identity condition) to fulfill its definition. \(\Box\)

For example, the vertices in the 20-cycle to the right in Figure 3 of Section 12 below can be rewritten as follows in their shown disposition, where \(a = 00011\) and \(b = 11101:\)

\[
\begin{align*}
\Upsilon((3,0),a) & \ Upsilon((3,1),a) \ Upsilon((4,0),a) \ Upsilon((4,1),a) \ Upsilon((0,0),a) \ Upsilon((0,1),a) \ Upsilon((1,0),a) \ Upsilon((1,1),a) \ Upsilon((2,0),a) \ Upsilon((2,1),a) \\
\Upsilon((3,0),b) & \ Upsilon((3,1),b) \ Upsilon((4,0),b) \ Upsilon((4,1),b) \ Upsilon((0,0),b) \ Upsilon((0,1),b) \ Upsilon((1,0),b) \ Upsilon((1,1),b) \ Upsilon((2,0),b) \ Upsilon((2,1),b)
\end{align*}
\]

representing a Hamilton cycle in \(M_k\), for \(k = 2\), invariant under the action of \(D_{2n}\).

Let the graph map \(\gamma_k : M_k/\pi \to R_k\) be the corresponding projection, as represented for \(k = 2\) in Figure 1. Then the canonical projection \(\rho_{D_{2n}} : M_k \to R_k\) is the composition of the canonical projection \(\rho_{Z_2} : M_k \to M_k/\pi\) with \(\gamma_k\). We remark that \(\Upsilon\) is regular just when \(M_k\) is taken as a directed graph, because (undirected) edges of \(M_k\) leading to loops of \(R_k\) via \(\rho_{D_{2n}}\) appear that are fixed by \(\Upsilon\). For example \(R_2\), represented as the image of the graph map \(\gamma_2\) depicted in Figure 1, contains two vertices and just one (vertical) edge between them, where each vertex is incident to two loops. The representation of \(M_2/\pi\) on its left has its edges indicated with colors 0,1,2. Here, the edge \(\epsilon = (11000, 11100)\) is fixed via \(\mathbb{N}\); not so for the each one of the two arcs composing \(\epsilon\).

In general, each vertex \(v\) of \(L_k/\pi\) will have its incident edges indicated with lexical colors 0,1,\ldots,\(k\) obtained by the following procedure arising from [13], so that \(L_k/\pi\) admits a \((k+1)\)-edge-coloring with color set \([k+1] = \{0,\ldots,k\}\).

## 9 Kierstead-Trotter lexical procedure

For each \(v \in L_k/\pi\) there are \(k + 1\) \(n\)-vectors of the form \(b_0b_1 \cdots b_{n-1} = 0b_1 \cdots b_{n-1}\) that represent \(v\) with \(b_0 = 0\). For each such \(n\)-vector, we take a grid \(\Gamma = P_{k+1} \sqcap P_{k+1}\) ([3], pg. 29), where \(P_{k+1}\) is the subgraph of the unit-distance graph of the real line \(\mathbb{R}\) induced by the set \([k+1] \subset \mathbb{Z} \subset \mathbb{R}\). We trace the diagonal \(\Delta\) of \(\Gamma\) from vertex \((0,0)\) to vertex \((k,k)\). (For \(k = 2\), \(\Delta\) is represented by a dashed line in the instances of Figure 2, analyzed up in Section 10). Recall that an arc of \(\Gamma\) is an ordered pair of adjacent vertices of \(\Gamma\). Based on [13], we build a directed \(2k\)-path \(D\) in \(\Gamma\) from \(w = (0,0)\) to \(w' = (k,k)\) in \(2k\) steps indexed from \(i = 0\) to \(i = 2k - 1\), as follows. Initially, set \(i = 0\) and \(w = (0,0)\) and let \(D\) be the 0-path containing solely \(w\). Repeat the following loop, formed subsequently by items (1)-(3), \(2k\) times:

1. If \(b_i = 0\) (resp., \(b_i = 1\)), then set \(w' := w + (1,0)\) (resp., \(w' := w + (0,1)\)).

2. Augment the \(i\)-path \(D\) by means of the arc \((w, w')\) of \(\Gamma\) into an \((i+1)\)-path, again denoted \(D\); in other words, reset \(D := D \cup (w, w')\); subsequently, set \(i := i + 1\) and \(w := w'\).

3. If \(w \neq (k,k)\), or equivalently, if \(i < 2k\), then go back to item (1).

4. Set \(\bar{v} \in L_{k+1}/\pi\) as a vertex of \(M_k/\pi\) adjacent to \(v \in L_k/\pi\) and obtained from a representative \(b_0b_1 \cdots b_{n-1} = 0b_1 \cdots b_{n-1}\) of \(v\) by replacing the entry \(b_0\) of \(v\) by \(\bar{b}_0 = 1\) in \(\bar{v}\), keeping the entries \(b_i\) of \(v\) with \(i > 0\) unchanged in \(\bar{v}\).
(5) Set the color of the edge $v\bar{v}$ to be the number $c$ of horizontal (alternatively vertical) arcs of $D$ below the diagonal $\Delta$ of $\Gamma$.

We remark that [13] highlighted the number $k + 1 - c$, where $c$ varies in $[k + 1]$, instead of establishing a well-defined 1-1 correspondence between $[k + 1]$ and the set of edges incident to $v$ in $L_k/\pi$, as we do here. In fact, if addition and subtraction in $[n]$ are taken modulo $n$ and we write $[y, x] = \{y, y + 1, y + 2, \ldots, x - 1\}$, for $x, y \in [n]$, and $S^c = [n] \setminus S$, for $S = \{i \in [n] : b_i = 1\} \subseteq [n]$, then the cardinalities of the sets $\{y \in S^c \setminus x : |[y, x] \cap S| < |[y, x] \cap S^c|\}$ yield all the numbers $k + 1 - c$ in 1-1 correspondence with our colors $c$, where $x \in S^c$ varies.

As in [13], the lexical procedure (or LP for short) just presented yields 1-factorizations of $R_k$, $M_k/\pi$ and $M_k$ by means of the edge colors $c = 0, 1, \ldots, k$. This lexical approach is compatible with the graphs $M_k/\pi$ and $R_k$, because each edge $e$ of $M_k$ has the same lexical color in $[k + 1]$ for both arcs composing $e$.

10 Ascending castling

In what follows, a color notation $\delta(v)$ is set for each vertex $v$ in $L_k/\pi$. In fact, there exists a unique $k$-word $\alpha = \alpha(v)$ with $[F(\alpha)] = \delta(v)$. We start by representing the lexical color assignment suggested in Figure 2 for $k = 2$, with the LP indicated by arrows “⇒” departing from $v = [00011]$ (top) and $v = [0010]$ (bottom) then going to the right via depiction of working sketches of $V(\Gamma)$ (separated by plus signs “+”) for each of the three representatives $b_0b_1 \cdots b_{n-1} = 0b_1 \cdots b_{n-1}$ (shown as a subtitle for each sketch, with the entry $b_0 = 0$ underscored) in which to trace the arcs of $D \subset \Gamma$ below $\Delta$, and finally pointing, via an arrow “→” departing from the representative $b_0b_1 \cdots b_{n-1} = 0b_1 \cdots b_{n-1}$ in each sketch subtitle, the number of horizontal arcs of $D$ below $\Delta$. Only arcs of $D \subset \Gamma$ are traced on each sketch of $V(\Gamma)$, with those below $\Delta$ indicated as darts in bold trace, and the remaining ones as segments in thin trace.

In each of the two cases in Figure 2, to the right of the three sketches, an arrow “⇒” points to an unparenthesized modification of the notation $(b_0b_1 \cdots b_{n-1})$ of $v$ obtained by setting as a subindex of each entry 0 the color obtained from its corresponding sketch, and an asterisk “∗” for each entry 1. Further to the right of this subindexed version of $v$, another arrow ”⇒” points to the string of length $n$ formed solely by the just established subindexes in the order they appear from left to right. This final notation is indicated by $\delta(v)$. For each such $\delta(v)$, there is a unique $k$-word $\alpha = \alpha(v)$ with $(F(\alpha)) = \delta(v)$, a fact whose proof depends on
the inverse operation to descending castling in Section 4, that we may call *ascending castling* and that can be presented as follows:

Given an $n$-tuple $v$ in $L_k/\pi$, let $W^i = k(k-1) \cdots (k-i)$ be the maximal initial $(i+1)$-substring of $\delta(v) = \delta^0(v)$, where $0 \leq i \leq k$. If $i = k$, then let $\alpha(v) = [00 \cdots 0]$ consist of $k-1$ zeros. Else, write $\delta^0(v) = [W^i|X|Y|Z^i]$, where $Z^i$ is a $j_0$-substring with $j_0 = i + 1$, and $X, Y$ start respectively at two integers $\ell$ and $\ell + 1 \leq k - i$. Let $\delta^1(v) = [W^i|X|Y|Z^i]$. If $\delta^1(v) = [k(k-1) \cdots 10 \cdots 1^*]$, then let $\alpha(v)$ differ from $\alpha_{k-1} \cdots \alpha_1 = 00 \cdots 0$ just by unit incrementation of $a_{j_0}$. Else, repeat the procedure above starting at $\delta^1(v)$, and so on. In the end, we obtain a finite sequence $\delta^0(v), \delta^1(v), \ldots, \delta^s(v)$ of $n$-tuples in $L_k/\pi$ with parameters $j_0 \geq j_1 \geq \ldots \geq j_s$ and $k$-words $\alpha(v^0), \alpha(v^1), \ldots, \alpha(v^s) = 00 \cdots 0$ and obtain $\alpha(v) = \alpha(v^0)$ from $\alpha^s(v) = 00 \cdots 0 = a_{k-1} \cdots a_1$ by unit incrementation of each $a_{j_i}$, for $i = 0, \ldots, s$, with each such incrementation yielding the corresponding $\alpha(v^i)$. Now, observe that $F$ is a bijection between the set of $k$-words and the set $L_k/\pi$, both of cardinality $C_k$. This shows that in order to work with $V(R_k)$ is enough to deal with the set of $k$-words, a fact useful in interpreting Theorem 5 below.

Take for example $\delta^0(v) = [40 * 1 * 2 * 3*] = [40 * 1 * 2 * 3*]$. Then,

$$
\begin{align*}
$$

yielding $\alpha(v) = \alpha(v^0) = 123$, with the $\alpha(v^i)$s corresponding to the formed rows being $\alpha(v^1) = [122], \alpha(v^2) = [121], \alpha(v^3) = [120], \alpha(v^4) = [110], \alpha(v^5) = [100]$, and $\alpha(v^6) = [000]$.

Thus, the function $F$ that sends the $k$-words onto their corresponding $n$-tuples $F(\alpha)$ in Section 4 happens to provide a backbone relating the succeeding applications of the LP to the elements of $L_k/\pi$, finally covering all of $L_k/\pi$. A pair of skew edges $(B_A)\mathcal{N}_\pi((B_A'))$ and $(B_A')\mathcal{N}(B_A)$ in $M_k/\pi$ is said to be a *skew reflective edge pair*. This provides a color notation for any $v \in L_{k+1}/\pi$ such that in each particular edge class mod $\pi$:

1. each edge receives the same color regardless of the endpoint on which the LP or its modification for $v \in L_{k+1}/\pi$ is applied;

2. each skew reflective edge pair in $M_k/\pi$ is assigned a sole color in $[k+1].$

The modification in item (1) consists in replacing in Figure 2 each $v$ by $\mathcal{N}_\pi(v)$ so that on the left we have now instead (00111) (top) and (01011) (bottom) with respective sketch subtitles

\[
\begin{align*}
00111 \rightarrow 2, & \quad 10101 \rightarrow 2, \\
01011 \rightarrow 2, & \quad 11001 \rightarrow 0, \\
01101 \rightarrow 0, & \quad 01101 \rightarrow 1.
\end{align*}
\]

resulting in similar sketches when the rules of the LP are taken with right-to-left reading-and-processing of the entries on the left side of the subtitles (before the arrows "→"), where now the values of each $b_i$ must be taken complemented.

Since a skew reflective edge pair in $M_k$ determines a unique edge $\epsilon$ of $R_k$ (and vice versa), the color received by this pair can be attributed to $\epsilon$, too. Clearly, each vertex of $M_k$ or $M_k/\pi$ or $R_k$ defines a bijection between its incident edges and the color set $[k+1]$. The edges obtained via $\mathcal{N}$ or $\mathcal{N}_\pi$ from these edges have the same corresponding colors because of the LP.
Theorem 5. A 1-factorization of $M_k/\pi$ formed by the edge colors $0,1,\ldots,k$ is obtained via the LP. This 1-factorization can be lifted to a covering 1-factorization of $M_k$ and can further be collapsed onto a folding 1-factorization of $R_k$ which induces a color notation $\delta(v)$ on each of its vertices $v$. Moreover, for each $v \in V(R_k)$ and induced notation $\delta(v)$, there is a unique $k$-word $\alpha = \alpha(v)$ such that $[F(\alpha)] = \delta(v)$.

Proof. As pointed out in item (2) above, each skew reflective edge pair in $M_k/\pi$ has its edges with the same color in $[k+1]$. Thus, the $[k+1]$-coloring of $M_k/\pi$ induces a well-defined $[k+1]$-coloring of $R_k$. This yields the claimed collapsing to a covering 1-factorization of $R_k$. The lifting to a covering 1-factorization in $M_k$ is immediate. The arguments above in this section and from Section 4 determine that the collapsing 1-factorization in $R_k$ induces the $k$-word $\alpha(v)$ claimed in the statement. \hfill \Box

Corollary 6. $L_k/\pi$ and $L_{k+1}/\pi$ can be represented respectively by the resulting classes $(F(A))$ and $(\delta(F(A)))$.

Proof. The corollary follows from Theorem 5 and its preceding discussion. \hfill \Box

11 Lexically colored adjacency table

From now on and justified by Theorem 5, we use the color notation $\delta(v)$ for the vertices $v$ of $R_k$ with no enclosures in parentheses or brackets as above. Furthermore, we consider a lexically colored adjacency table for $R_k$ in which the vertices $F(\alpha)$ of $R_k$ are expressed via their notation $\alpha$, and with the order of such $\alpha$s taken stair-wise, as agreed before Observation 1. According to this, we view $R_k$ as the graph whose vertices are the $k$-words $\alpha$ and whose adjacency is inherited from that of their $\delta$-notation in $R_k$ via pullback by $F^{-1}$ (namely, via ascending castling). In writing elements of $R_k$, we avoid now parentheses or brackets.

In Table III, examples of such disposition are shown for $k = 2$ and 3. Notice that the neighbors of each $F(\alpha)$ in the second column are presented as $F^0(\alpha)$, $F^1(\alpha)$, $\ldots$, $F^k(\alpha)$ respectively for the colors $0,1,\ldots,k$ of the edges incident to them, where the notation is given via the direct effect of the function $\delta$. The last four columns yield the $k$-words $\alpha^0$, $\alpha^1$, $\ldots$, $\alpha^k$ associated via $F^{-1}$ respectively with the listed neighbor vertices $F^0(\alpha)$, $F^1(\alpha)$, $\ldots$, $F^k(\alpha)$ of $F(\alpha)$ in $R_k$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$F(\alpha)$</th>
<th>$F^0(\alpha)$</th>
<th>$F^1(\alpha)$</th>
<th>$F^2(\alpha)$</th>
<th>$F^3(\alpha)$</th>
<th>$\alpha^0$</th>
<th>$\alpha^1$</th>
<th>$\alpha^2$</th>
<th>$\alpha^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>210 **</td>
<td>210 **</td>
<td>20 * 1*</td>
<td>10 ** 2*</td>
<td>-</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>20 * 1*</td>
<td>1 * 20*</td>
<td>210 **</td>
<td>0 * 1* 2</td>
<td>-</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-</td>
</tr>
<tr>
<td>00</td>
<td>3210****</td>
<td>3210****</td>
<td>320<em>1**2</em></td>
<td>310<em>2</em></td>
<td>210***3</td>
<td>00</td>
<td>10</td>
<td>01</td>
<td>00</td>
</tr>
<tr>
<td>01</td>
<td>310<em>2</em></td>
<td>2*310**</td>
<td>2<em>30</em>1*</td>
<td>3210***</td>
<td>1*20**3</td>
<td>01</td>
<td>12</td>
<td>00</td>
<td>11</td>
</tr>
<tr>
<td>10</td>
<td>320*1**</td>
<td>31*20**</td>
<td>3210***</td>
<td>30<em>1</em>2*</td>
<td>20<em>1</em>**3</td>
<td>11</td>
<td>00</td>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>11</td>
<td>31*20**</td>
<td>320*1**</td>
<td>20<em>1**31</em></td>
<td>31*20**</td>
<td>10<em>2</em>3</td>
<td>10</td>
<td>11</td>
<td>11</td>
<td>01</td>
</tr>
<tr>
<td>12</td>
<td>30<em>1</em>2*</td>
<td>1<em>2</em>30*</td>
<td>2*310**</td>
<td>320*1**</td>
<td>0<em>1</em>2*3</td>
<td>12</td>
<td>01</td>
<td>10</td>
<td>12</td>
</tr>
</tbody>
</table>
For \( k = 4 \), observe in Table IV the resulting stair-wise adjacency disposition. In general, for any \( k > 1 \), the columns \( \alpha^i \) of the stair-wise adjacency table preserve their respective \( j \)-th entries (taken from right to left, so \( j = k, k - 1, \ldots, 2, 1 \)) in the following way: \( j(\alpha^0) = k \), \( j(\alpha^2) = k \), \( j(\alpha^3) = k - 1 \), \ldots, \( j(\alpha^{k-1}) = 2 \), \( j(\alpha^k) = 1 \), while we do not have such a simple entry invariance rule for column \( \alpha^1 \). A further analysis of the relation between \( \alpha \) and each of \( \alpha^1, \ldots, \alpha^{k-1} \) proceeds in [8], Sections 2-7, in terms of \( k \)-words, as in Table IV.

### Table IV

| \( \alpha \) | \( \alpha^0 \) | \( \alpha^1 \) | \( \alpha^2 \) | \( \alpha^3 \) | \( \alpha^4 \) |
|---|---|---|---|---|
| 000 | 000 | 000 | 000 |
| 001 | 001 | 010 | 011 |
| 010 | 011 | 010 | 112 |
| 011 | 010 | 112 | 110 |
| 012 | 012 | 012 | 120 |
| 013 | 013 | 013 | 123 |
| 100 | 100 | 100 | 100 |
| 101 | 101 | 101 | 121 |
| 102 | 102 | 102 | 123 |
| 103 | 103 | 103 | 123 |
| 110 | 110 | 110 | 111 |
| 111 | 111 | 111 | 112 |
| 112 | 112 | 112 | 112 |
| 120 | 120 | 120 | 120 |
| 121 | 121 | 121 | 121 |
| 122 | 122 | 122 | 122 |
| 123 | 123 | 123 | 123 |

For every \( k > 1 \), each color defines an involution, as displayed in Table V, where fixed points are enclosed in parentheses, the remaining cycles are all transpositions and such fixed points and transpositions between columns \( \alpha \) and \( \alpha^i \) in the tables above are presented after a corresponding header “i.”

### Table V

For \( k = 2 \), \( \xi_2 \) first into a Hamilton cycle \( \eta_2 \) of \( M_2/\pi \) invariant under \( \Upsilon \) (Section 8), thus constituting a Hamilton cycle placed in a dihedrally symmetric fashion in \( M_k \), meaning it is invariant under the dihedral-group action.

First, they pull back \( \xi_k \) via the inverse image \( \gamma_k^{-1} \) (of \( \gamma_k \) in Section 8) onto a Hamilton cycle \( \zeta_k \) in \( M_k/\pi \) invariant under \( \Upsilon' \) (Section 6), where a loop at each end of \( \xi_k \) lifts onto its corresponding parallel edge in \( M_k/\pi \). Second, they pull \( \zeta_k \) back via \( \rho_{\pi_n}^{-1} \) onto a \( \eta_k \) in \( M_k \) invariant under \( \Upsilon \).

For example, the reflection about \( \ell \) on the left of Figure 3 is used to transform \( \xi_2 \) first into a Hamilton cycle \( \zeta_2 \) of \( M_2/\pi \) invariant under the action of \( \mathbb{Z}_2 \) induced by \( \mathfrak{N} \), represented

12 Shields-Savage paths

Inspired in the construction technique considered by the author and his students in [6, 7], Shields and Savage showed in their Lemma 3 [18] that a Hamilton path \( \xi_k \) in \( R_k \) starting at \( [F(0^{k-1})] = [0^{k+1}] \) and ending at \( [F(12 \ldots k)] = [0(01)^k] \) exists that determines a Hamilton cycle \( \eta_k \) in \( M_k \) invariant under \( \Upsilon \) (Section 8), thus constituting a Hamilton cycle placed in a dihedrally symmetric fashion in \( M_k \), meaning it is invariant under the dihedral-group action.
on the figure, and then into a path of length $2|V(R_2)| = 4$ starting at $00101 = x^2 + x^4$ and ending at $01010 = x + x^3$, in the same class mod $1 + x^5$, that can be repeated five times to form a Hamilton cycle $\eta_2$ invariant under $\Upsilon$, represented on the rest of the figure.

\[\text{Figure 3: Hamilton cycles in } M_2/\pi \text{ and } M_2\]

In the same way and because of Lemma 3 of [18], a Hamilton cycle $\eta_k$ in $M_k$ invariant under the action of $\mathbb{Z}_2$ is guaranteed by the determination of a Hamilton path $\xi_k$ in $R_k$ from vertex $\delta(0^{k+1}1^k) = k(k - 1) \cdots 21 \cdots *$ to vertex $\delta(0(10)^k) = k0 * 1 * 2 \cdots * (k - 1)*$. A Shields-Savage path $\xi_k$ offers the finite sequence of colors of successive edges in $\xi_k$ as a succinct code for the Hamilton cycle $\eta_k$. By describing $\xi_k$ via the sequence of its edge colors, or admissible color sequence, Table VI shows for $k = 2, 3$ such sequences, namely: $c_{\alpha_0}^2 = 1$ and $c_{\alpha_0}^3 c_{\alpha_0}^3 c_{\alpha_0}^3 c_{\alpha_0}^3 = 1031$, where $i_0$ is the order in which the color $c_{\alpha_0}^k$ is selected.

\[\text{Table VI}\]

<table>
<thead>
<tr>
<th>$i_0$</th>
<th>$\alpha_0$</th>
<th>$F(\alpha_0)$</th>
<th>$\psi(\alpha_0)$</th>
<th>$\xi(\alpha_0)$</th>
<th>$\eta(\psi(\alpha_0))$</th>
<th>$\eta(\xi(\alpha_0))$</th>
<th>$c_{\alpha_0}^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i_0$</td>
<td>$0$</td>
<td>$210**$</td>
<td>$00101$</td>
<td>$0, 1, 0, 1 , 0, 1$</td>
<td>$0, 0, 0, 1, 0, 1, 1$</td>
<td>$00111$</td>
<td>$01$</td>
</tr>
<tr>
<td>$20$</td>
<td>$00$</td>
<td>$3210**$</td>
<td>$001011$</td>
<td>$0, 0, 1 , 0, 1, 1, 0, 1$</td>
<td>$0, 0, 0, 1, 0, 1, 1$</td>
<td>$001011$</td>
<td>$0$</td>
</tr>
<tr>
<td>$00$</td>
<td>$10$</td>
<td>$3201**$</td>
<td>$001011$</td>
<td>$0, 0, 1 , 0, 1, 1, 0, 1$</td>
<td>$0, 0, 0, 1, 0, 1, 1$</td>
<td>$001011$</td>
<td>$0$</td>
</tr>
<tr>
<td>$01$</td>
<td>$12$</td>
<td>$301*+2*$</td>
<td>$1010011$</td>
<td>$1, 0, 0, 1, 1, 0, 1, 0, 1$</td>
<td>$1, 0, 0, 1, 1, 0, 1, 0, 1$</td>
<td>$101101$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

In fact, the feasible admissible color sequences for $k = 2, 3, 4$ are (lexicographically order from top to bottom and then from left to right) as in Table VII.

\[\text{Table VII}\]

<table>
<thead>
<tr>
<th>$k=2$</th>
<th>$1$</th>
<th>$k=4$</th>
<th>$1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k=3$</td>
<td>$1031$</td>
<td>$1203101313131$</td>
<td>$1203101313131$</td>
</tr>
<tr>
<td>$2302$</td>
<td>$1203101313131$</td>
<td>$1203101313131$</td>
<td></td>
</tr>
</tbody>
</table>

Indicated here with a sign $\pm$ to the right are the admissible color sequences for which the corresponding sequences of succeeding $k$-words form a maximum (+) and a minimum (−) according to the stair-wise order of the $k$-words involved lexicographically in the sequences. This yields a different notion of extremality for the admissible color sequences. For $k = 3, 4$, we present both these extremal objects as follows:

<table>
<thead>
<tr>
<th>$k=2$</th>
<th>$1$</th>
<th>$k=4$</th>
<th>$1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k=3$</td>
<td>$1031$</td>
<td>$1203101313131$</td>
<td>$1203101313131$</td>
</tr>
<tr>
<td>$2302$</td>
<td>$1203101313131$</td>
<td>$1203101313131$</td>
<td></td>
</tr>
</tbody>
</table>

| $00 (1) 10 (0) 11 (3) 01 (1) 12 +$ |
| $00 (2) 01 (3) 11 (0) 10 (2) 12 −$ |
| $00 (3) 001 (4) 011 (3) 111 (1) 110 (3) 012 (4) 122 (1) 112 (3) 010 (1) 121 (4) 101 (3) 100 (2) 120 (3) 123 −$ |

| $00 (1) 10 (0) 11 (3) 01 (1) 12 +$ |
| $00 (2) 01 (3) 11 (0) 10 (2) 12 −$ |
| $00 (3) 001 (4) 011 (3) 111 (1) 110 (3) 012 (4) 122 (1) 112 (3) 010 (1) 121 (4) 101 (3) 100 (2) 120 (3) 123 −$ |
where succeeding $k$-words are separated by adjacency colors expressed between parentheses, so that by concatenating from left to right the contents of those parentheses yields the corresponding admissible color sequence. On the other hand, the minimum and maximum of the admissible color sequence for $k = 5$ are

10102012010043010201034213101010121;
4545354535453545354535453545354543.

Extremality in the second sense above yields for $k = 5$ the contents of Table VIII.

<table>
<thead>
<tr>
<th>TABLE VIII</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000 (1) 1000 (2) 1200 (3) 1230 (0) 1233 (4) 1231 (0) 1232 (3)</td>
</tr>
<tr>
<td>1221 (5) 1211 (0) 1222 (1) 1120 (4) 1123 (5) 1223 (3) 1212 (4)</td>
</tr>
<tr>
<td>1010 (5) 1210 (3) 1220 (1) 1122 (4) 1121 (3) 0121 (2) 0010 (1)</td>
</tr>
<tr>
<td>1011 (5) 1001 (2) 1201 (1) 0112 (3) 1112 (4) 1110 (0) 1111 (3)</td>
</tr>
<tr>
<td>0122 (0) 0120 (3) 1100 (4) 0101 (2) 0001 (5) 0011 (4) 0111 (2)</td>
</tr>
<tr>
<td>0110 (0) 0100 (4) 1101 (3) 0123 (2) 0012 (1) 1012 (2) 1234 +</td>
</tr>
<tr>
<td>0000 (4) 0001 (5) 0011 (0) 0010 (5) 0110 (4) 0012 (5) 0122 (2)</td>
</tr>
<tr>
<td>0112 (0) 0101 (4) 1100 (5) 0100 (4) 1101 (5) 0121 (3) 1121 (0)</td>
</tr>
<tr>
<td>1010 (3) 1000 (4) 1001 (5) 1011 (0) 1120 (4) 1123 (0) 1012 (4)</td>
</tr>
<tr>
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References


