# A construction based on the Biggs-Smith graph

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**Abstract**

Departing from the Biggs-Smith graph, we construct a connected 12-regular 102-vertex graph $Y$ of diameter 3 that can be presented as the edge-disjoint union of 102 copies of $K_4$ or as the square-disjoint union of 102 cuboctahedra $L(Q_3)$, with each edge shared by four $L(Q_3)$s and each triangle shared by two $L(Q_3)$s and a sole copy of $K_4$. Moreover, $Y$ is the non-line-graphical Menger graph of a self-dual $(102_4)$-configuration. Furthermore, $Y$ is a $\{K_4, L(Q_3)\}_{K_3}$-ultrahomogeneous graph.

## 1 Introduction

Ultrahomogeneous (or UH) graphs (resp. digraphs) go back to [16], [11], [15], [3] and [12], (resp. [10], [14] and [6]), but for our construction, $\mathcal{C}$-UH graphs are needed: Given a collection $\mathcal{C}$ of (di)graphs closed under isomorphisms, a (di)graph $G$ is *$\mathcal{C}$-UH* if every isomorphism between two induced members of $\mathcal{C}$ in $G$ extends to an automorphism of $G$. If $\mathcal{C} = \{H\}$ is the isomorphism class of a graph $H$, then $G$ is said to be $\{H\}$-UH or $H$-UH. In [13], $\mathcal{C}$-UH graphs are defined and studied when $\mathcal{C}$ is the collection formed either by: **(a)** the complete graphs, or **(b)** the disjoint unions of complete graphs, or **(c)** the complements of those disjoint unions. In [8], a $\{K_4, K_{2,2,2}\}$-UH graph $X$ that fastens copies of $K_4$ and $K_{2,2,2}$, (objects in (a) and (c) above, resp.), was presented. In this note, we construct a graph $Y$ that stands in contrast to the graph of [8].

We may consider a graph $G$ as a digraph by taking each edge $e$ of $G$ as a pair of oppositely oriented (or O-O) arcs $\vec{e}$ and $(\vec{e})^{-1}$. Then, 'zipping' $\vec{e}$ and $(\vec{e})^{-1}$ allows to recover $e$, a technique to be used below. (In [9], however, a strongly connected $C_4$-UH oriented graph without O-O arcs was presented).

Let $M$ be a sub(di)graph of a (di)graph $H$ and let $G$ be both an $M$-UH and an $H$-UH (di)graph. Then, $G$ is an *$(H;M)$-UH (di)graph* if, given a copy $H_0$ of $H$ in $G$ containing a copy $M_0$ of $M$, there exists exactly one copy $H_1 \neq H_0$ of $H$ in $G$ with $V(H_0) \cap V(H_1) = V(M_0)$ and $A(H_0) \cap \bar{A}(H_1) = A(M_0)$, where $\bar{A}(H_1)$ is formed by those arcs $(\vec{e})^{-1}$ whose orientations are reversed with respect to the orientations of the arcs $\vec{e}$ of $A(H_1)$, and moreover: no more vertices or arcs than those in $M_0$ are shared by $H_0$ and $H_1$. We may say that such a $G$ is *tightly* *fastened*. The directed case is used in the construction of $Y$. In the undirected case, the vertex and arc conditions above can be condensed as $H_0 \cap H_1 = M_0$, generalized by saying that, for fixed $0 < \ell \in \mathbf{Z}$, an $(H;M)$-UH graph $G$ is an *$\ell$-fastened* $(H;M)$-UH graph if, given a copy $H_0$ of $H$ in $G$ containing a copy $M_0$ of $M$, there exist exactly $\ell$ copies $H_i \neq H_0$ of $H$ in $G$ such that $H_i \cap H_0 = M_0$, for each one of $i = 1, 2, \ldots, \ell$, and such that no more vertices or edges than those in $M_0$ are shared by each two of $H_0, H_1, \ldots, H_\ell$.

To better restate the cited result from [8], let $H_i$ be a connected graph, for $i = 1, 2$, with $H_i \not\subset H_j$, $(i \neq j \in \{1,2\})$. A graph $G$ is an *$\{H_2, H_1\}_{K_2}$-UH graph* if for each $i \in \{1,2\}$: **(a)** $G$ is an $H_i$-UH graph; **(b)** $G$ is representable as an edge-disjoint union of a number $n_i$ of induced copies of $H_i$; **(c)** $G$ has a constant number $m_i$ of copies of $H_i$ incident at each vertex, with no two such copies sharing more than one vertex; **(d)** $G$ has exactly $n_i$ copies of $H_i$ as induced subgraphs; **(e)** $G$ has each edge in exactly one copy of $H_i$. We may say that such a $G$ is *tightly $K_2$-fastened*.

In [8], it is shown that $X$ is a $\{K_4, K_{2,2,2}\}_{K_2}$-UH graph; in fact, this is the first known such $\mathcal{C}$-UH graph that is not a line graph, a fact relevant in view, for example, of the line graphs of the $d$-cubes $Q_d$, ($3 \leq d \in \mathbf{Z}$, say the cuboctahedron $L(Q_3)$), which are $\{K_d, K_{2,2}\}_{K_2}$-UH. In addition, $X$ has exactly 42 copies of $K_4$ and 21 copies of $K_{2,2,2}$ as subgraphs. $X$ also has exactly four copies of $K_4$ and three copies of $K_{2,2,2}$ incident at each one of its 42 vertices. Considering self-dual configurations and their Menger graphs as in [7], $X$ is seen to be the Menger graph of a self-dual $(42_4)$-configuration.

To find a graph $Y$ that stands in contrast to $X$, we start with the Biggs-Smith graph $\mathcal{S}$, ([2, 5]). This is a distance-transitive hamiltonian cubic graph of order $n = 102$, diameter $d = 7$, girth $g = 9$, arc-transitivity $k = 4$, number of $g$-cycles $\eta = 136$ and number of automorphisms $a = 2448$.

Given a finite graph $H$ and a subgraph $M$ of $H$ with $|V(H)| > 3$, a graph $\Gamma$ is (strongly fastened) *SF $(H;M)$-UH* if there is a descending sequence of connected subgraphs $M = M_1, \ldots, M_{|V(H)|-2} \equiv K_2$ such that: **(a)** $M_{i+1}$ is obtained from $M_i$ by the deletion of a sole vertex, for $i = 1, \ldots, |V(H)| - 3$ and **(b)** $\Gamma$ is a $(2^i - 1)$-fastened $(H; M_i)$-UH graph, for $i = 1, \ldots, |V(H)| - 2$.

Let $P_k$ and $\vec{P}_k$ be respectively a $(k-1)$-path and a directed $(k-1)$-path (of lengths $k-1$). Let $C_g$ and $\vec{C}_g$ be respectively a cycle and a directed cycle of length $g$. Theorem 1 below asserts that $\mathcal{S}$ is SF $(C_g; P_k)$-UH, or SF $(C_9, P_4)$-UH.

The claimed graph $Y$ is another SF $(H;M)$-UH graph (see Theorem 4) but some additional concepts are needed. A graph $G$ is *$rK_s$-frequent* if every edge $e$ of $G$ is the intersection in $G$ of exactly $r$ copies of $K_s$, and these copies have only $e$ and its endvertices in common. (For example, $K_4$ is $2K_3$-frequent, and $L(Q_3)$ is $1K_3$-frequent). A graph $G$ is *$\{H_2, H_1\}_{K_3}$-UH*, where $H_i$ is $iK_3$-frequent, $(i = 1, 2)$, if: **(a)** $G$ is an $H_2$-UH graph and an edge-disjoint union of copies of $H_2$; **(b)** $G$ is SF $(H_1; K_3)$-UH; **(c)** each copy of $H_2$ in $G$ has each of its copies of $K_3$ in common exactly with *two* copies of $H_1$ in $G$.

Given $0 < \ell \in \mathbf{Z}$ and a graph $C$ such that $\ell$ is at most the diameter of $C$, recall that the $\ell$-power graph $C^\ell$ of $C$ has $V(C^\ell) = V(C)$ and that two vertices are adjacent in $C^\ell$ if and only if they are at distance $\ell$ in $C$. Theorem 2 establishes that $\mathcal{S}$ is a $(\vec{C}_9;\vec{P}_4)$-UH digraph. Elevating the resulting oriented cycles to the third power (or cube) enables the construction, in Section 3, of $Y$ as a non-line-graphical $\{K_4,L(Q_3)\}_{K_3}$-UH graph (Theorems 3-4) via 'zipping' oriented 9-cycles along O-O 3-arcs given by $\mathcal{S}$ seen as a $(\vec{C}_9;\vec{P}_4)$-UH digraph. In particular, $\mathcal{S}$ yields $Y$ as the Menger graph of a self-dual $(102_4)$-configuration. Moreover, no two copies of $L(Q_3)$ in $Y$ have a 4-cycle in common, so they are *square-disjoint*. Furthermore, each edge of $Y$ is exactly in four copies of $L(Q_3)$.

## 2 Properties of the Biggs-Smith graph $\mathcal{S}$

**Theorem 1** *$\mathcal{S}$ is an SF $(C_g;P_{i+2})$-UH graph, for $i=0,1,\ldots,k-2=2$. In particular, $\mathcal{S}$ is a $(C_9;P_4)$-UH graph and has exactly $2^{k-2}3ng^{-1}=136$ 9-cycles.*

*Proof.* We have to see that $\mathcal{S}$ is a $(2^i-1)$-fastened $(C_9;P_{4-i})$-UH graph, for $i=0,1,\ldots,k-2=2$. In fact, each $(4-i-1)$-path $P=P_{4-i}$ of $\mathcal{S}$ is shared exactly by $2^i$ 9-cycles of $\mathcal{S}$, for $i=0,1,\ldots,k-2=2$. Moreover, each two of these $2^i$ 9-cycles have just $P$ in common. Now, the statement follows from a simple counting argument. □

**Theorem 2** *$\mathcal{S}$ is a $(\vec{C}_9;\vec{P}_4)$-UH graph.*

*Proof.* If $\mathcal{S}$ is $(\vec{C}_g;\vec{P}_k)$-UH, an assignment of an orientation to each $g$-cycle of $\mathcal{S}$ such that the two $g$-cycles shared by each $(k-1)$-path receive opposite orientations is called a $(\vec{C}_g;\vec{P}_k)$-*orientation assignment* (or $(\vec{C}_g;\vec{P}_k)$-OA).

A collection of $\eta$ oriented $g$-cycles corresponding to the $\eta=136$ $g$-cycles of $\mathcal{S}$, for a particular $(\vec{C}_g;\vec{P}_k)$-OA is called an $(\eta\vec{C}_g;\vec{P}_k)$-OAC (or $(\eta\vec{C}_g;\vec{P}_k)$-OA *collection*).

$\mathcal{S}$ can be reconstructed from four 17-cycles $y=A$, $D$, $C$, $F$, namely $A=(A_0,A_1,\ldots,A_g\}$, $D=(D_0,D_2,\ldots,D_f\}$, $C=(C_0,C_4,\ldots,C_d\}$, $F=(F_0,F_8,\ldots,F_9)$, (where each $y$ has vertices $y_i$ with $i$ taken as an heptadecimal subindex, up to $g=16$, and advancing in 1,2,4,8 units mod 17, stepwise from left to right, resp.), by adding a 6-vertex tree with degree-1 vertices $A_i,C_i,D_i,F_i$ and degree-2 vertices $B_i,E_i$ and containing the 3-paths $A_iB_iC_i$ and $D_iE_iF_i$, for each $i\in\mathbf{Z}_{17}$. Then, $\mathcal{S}$ admits the $(136\,\vec{C}_9;\vec{P}_4)$-OAC formed by the oriented 9-cycles

$$\begin{array}{ll}
S^0=(A_0A_1B_1C_1C_5C_9C_dC_0B_0), & W^0=(A_0A_1B_1E_1F_1F_9F_0E_0B_0),\\
T^0=(C_0C_4B_4A_4A_3A_2A_1A_0B_0), & X^0=(C_0C_4B_4E_4D_4D_2D_0E_0B_0),\\
U^0=(E_0F_0F_9F_1F_aF_2E_2D_2D_0), & Y^0=(E_0B_0A_0A_1A_2B_2E_2D_2D_0),\\
V^0=(E_0D_0D_2D_4D_6D_8E_8F_8F_0), & Z^0=(F_0F_8E_8B_8C_8C_4C_0B_0E_0),
\end{array}$$

and those obtained from these eight 9-cycles by adding $x\in\mathbf{Z}_{17}$ uniformly mod 17 to all subindices. These 9-cycles are denoted $S^x,T^x$, etc., where $x\in\mathbf{Z}_{17}$. □

## 3 The $\{K_4,L(Q_3)\}_{K_3}$-UH graph $Y$

Let $\mathcal{C}_9^3(\mathcal{S})$ be the collection of cubes of oriented 9-cycles of the $(136\,\vec{C}_9;\vec{P}_4)$-OAC of $\mathcal{S}$ in the proof of Theorem 2. Let us write $C_9$ for $\vec{C}_9$ and $C_9^3$ for $\vec{C}_9^3$. The initial vertex $w_0$, the initial flag, the terminal flag and the terminal vertex $w_1$ of each arc $\vec{e}=w_0w_1$ of a member $C_9^3$ of $\mathcal{C}_9^3(\mathcal{S})$ are indicated or marked pictorially, respectively, by the names of the vertices $v_0,v_1,v_2,v_3$ of the 3-arc $\vec{E}=v_0v_1v_2v_3$ in $C_9$ for which $\vec{e}$ stands in $C_9^3$. This allows to 'zip' the cycles $C_9^3$ along their O-O 3-arcs to obtain $Y$. The following transformations are performed:

$$\mathcal{S} \quad\rightarrow\quad (136\,\vec{C}_9;\vec{P}_4)\text{-OAC}(\mathcal{S}) \quad\rightarrow\quad \mathcal{C}_9^3(\mathcal{S}) \quad\rightarrow\quad Y.$$

The cubes $C_9^3$ of the 136 9-cycles $C_9$ of $\mathcal{S}$ are formed by three disjoint 3-cycles each, yielding a total of $3\times 136=408$ 3-cycles. In fact, the $(136\,\vec{C}_9;\vec{P}_4)$-OAC in the proof of Theorem 2 for $\mathcal{S}$ determines a $(408\,\vec{C}_3;\vec{P}_2)$-OAC for $Y$. The oriented 3-cycles in this $(408\,\vec{C}_3;\vec{P}_2)$-OAC are 'zipped' along the pairs of O-O copies of $\vec{P}_2$ obtained as cubes of O-O copies of $\vec{P}_4$ in $\mathcal{S}$. The resulting 'zipping' of O-O arcs yields 102 copies of $K_4$. These can be subdivided into six subcollections $\{y^i\}$ of 17 copies of $K_4$ each, where $y\in\{A,B,C,D,E,F\}$ and $i\in\{0,1,\dots,16=g\}=\mathbf{Z}_{17}$. The vertex sets $V(y^i)$ of these copies $y^i$ of $K_4$, each of them followed by the set $\Lambda(y_i)$ of copies of $K_4$ containing the corresponding vertex $y_i$ (as in the notation of the proof of Theorem 2), are as follows:

$$\begin{array}{llllllll}
V(A^x)=\{C_x, & D_x, & E_{x+4}, & E_{x-4}\}, & \Lambda(A_x)=\{C^x, & D^x, & E^x, & E^{x+3}\};\\
V(B^x)=\{D_{x-8}, & D_{x-1}, & F_x, & F_{x+7}\}, & \Lambda(B_x)=\{D^{x+2}, & D^{x-2}, & F^x, & F^{x-1}\};\\
V(C^x)=\{A_x, & F_x, & E_{x+2}, & E_{x-1}\}, & \Lambda(C_x)=\{A^x, & F^{x+8}, & E^{x+4}, & E^{x-1}\};\\
V(D^x)=\{A_x, & D_x, & B_{x+2}, & B_{x-2}\}, & \Lambda(D_x)=\{A^x, & D^x, & B^{x+2}, & B^{x+8}\};\\
V(E^x)=\{A_x, & A_{x-2}, & C_{x+1}, & C_{x-4}\}, & \Lambda(E_x)=\{A^{x+4}, & A^{x-4}, & C^{x+1}, & C^{x-1}\};\\
V(F^x)=\{C_{x-8}, & F_{x-8}, & B_x, & B_{x+1}\}, & \Lambda(F_x)=\{C^x, & F^{x+8}, & B^x, & B^{s-7}\};
\end{array}$$

where $x\in\mathbf{Z}_{17}$. This presentation emphasizes a duality existing between the 102 vertices of $\mathcal{S}$ and the 102 copies of $K_4$ just obtained. A corresponding duality map here is given by an isomorphism from $\mathcal{S}$, presented as in the proof of Theorem 2, onto a graph $\mathcal{S}'$ isomorphic to $\mathcal{S}$ and that can be obtained similarly from the four 17-cycles $A'=(A^0,A^3,\dots,A^e)$, $D'=(D^0,D^7,\dots,D^a)$, $C'=(C^0,C^c,\dots,C^5)$, $F'=(F^8,F^2,\dots,F^e)$ (which advance the vertex subindices in $3=1\times 3$, $7=2\times(-5)$, $12=4\times 3$, $11=8\times(-5)$ units mod 17 stepwise from left to right, respectively) by adding a 6-vertex tree with degree-one vertices $A^{3x},C^{3x},D^{-5x},F^{8-5x}$ and degree-two vertices $B^{5-7x},E^{10-6x}$ and containing the 3-paths $A^{3i}B^{5-7i}C^{3i}$ and $D^{-5x}E^{10-6x}F^{8-5x}$, for each $x\in\mathbf{Z}_{17}$.

The oriented 3-cycles that are components of the cubes of the 9-oriented cycles in the proof of Theorem 2 are:

$$\begin{array}{l}
S^0\rightarrow\{E^0\backslash A_e=(A_0,C_1,C_d),\ E^4\backslash A_4=(A_1,C_5,C_0),\ F^0\backslash F_9=(B_1,C_9,B_0)\};\\
T^0\rightarrow\{E^4\backslash C_5=(C_0,A_4,A_1),\ E^3\backslash C_g=(C_4,A_3,A_0),\ D^2\backslash D_2=(B_4,A_2,B_0)\};\\
U^0\rightarrow\{C^1\backslash A_1=(E_0,F_1,E_2),\ B^a\backslash D_8=(F_0,F_a,D_2),\ B^2\backslash D_b=(F_9,F_2,D_0)\};\\
V^0\rightarrow\{A^4\backslash C_4=(E_0,D_4,E_8),\ B^8\backslash F_f=(D_0,D_6,F_8),\ B^a\backslash F_a=(D_2,D_8,F_0)\};\\
W^0\rightarrow\{C^0\backslash E_g=(A_0,E_1,F_0),\ C^1\backslash E_2=(A_1,F_1,E_0),\ F^0\backslash C_9=(B_1,F_9,B_0)\};\\
X^0\rightarrow\{A^0\backslash E_d=(C_0,E_4,D_0),\ A^4\backslash E_8=(C_4,D_4,E_0),\ D^2\backslash A_2=(B_4,D_2,B_0)\};\\
Y^0\rightarrow\{C^1\backslash F_1=(E_0,A_1,E_2),\ D^2\backslash B_4=(B_0,A_2,D_2),\ D^0\backslash B_f=(A_0,B_2,D_0)\};\\
Z^0\rightarrow\{F^8\backslash B_9=(F_0,B_8,C_0),\ F^g\backslash B_g=(F_8,C_8,B_0),\ A^4\backslash D_4=(E_8,C_4,E_0)\}.
\end{array}$$

Because of the properties of $\mathcal{S}$ in Section 2, it can be seen that $Y$ is a $K_4$-UH graph. Moreover, the vertices and copies of $K_4$ in $Y$ are the points and lines of a self-dual $(102_4)$-configuration, which in turn has $Y$ as its Menger graph. (Compare with [7, 8]). However, in view of Beineke's characterization of line graphs [1], and observing that $Y$ contains induced copies of $K_{1,3}$, which are forbidden for line graphs of simple graphs, we conclude that $Y$ is non-line-graphical, as commented above for the Menger graph of the self-dual $(42_4)$-configuration treated in [8]. We obtain the following statement (via Magma).

**Theorem 3** *$Y$ is an edge-disjoint union of 102 copies of $K_4$, with four such copies incident to each vertex. Moreover, $Y$ is a non-line-graphical $K_4$-UH graph. Its vertices and copies of $K_4$ are the points and lines, respectively, of a self-dual $(102_4)$-configuration, which in turn has $Y$ as its Menger graph. This is an arc-transitive graph with regular degree 12, diameter 3, distance distribution $(1, 12, 78, 11)$ and automorphism-group order 2448. Its associated Levi graph [7] is a 2-arc-transitive graph with regular degree 4, diameter 6, distance distribution $(1, 4, 12, 36, 78, 62, 11)$ and automorphism-group order 4896.* $\square$

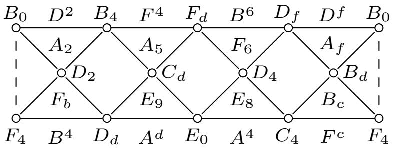

Figure 1: Copy $\overline{A^0}$ of $L(Q_3)$ in $Y$

Each of the 102 copies of $K_4$ in $Y$ arises from the cubes of four of the 136 9-cycles of $\mathcal{S}$. The subgraph of $\mathcal{S}$ spanned by these four 9-cycles contains four degree-three vertices, (heads and tails of corresponding 3-arcs $\vec{E}$ as in the first paragraph of this section), and twelve degree-two vertices, (middle vertices in those 3-arcs $\vec{E}$). These twelve vertices form a copy $\mathcal{L}$ of $L(Q_3)$ in $Y$. For the copy $A^0$ of $K_4$ in $Y$, the copy $\mathcal{L} = \overline{A^0}$ of $L(Q_3)$ in $Y$ can be represented as in Figure 1, where: **(a)** the leftmost and rightmost dotted lines are identified by parallel translation; **(b)** each of the eight shown copies of $K_3$ forms part of a corresponding copy of $K_4$ (among the 102 in $Y$) cited externally about its horizontal edge, with the fourth vertex cited internally. By presenting the elements of such a representation orderly, we may denote the copies $\overline{y^0}$ of $L(Q_3)$ as follows, for $y = A, B, C, D, E, F$:

$\overline{A^0}$: $(B_0B_4F_dD_f)$ $(F_4D_dE_0C_4)$ $D_2C_dD_4B_d$ $(D^2A_2F^4A_5B^6F_6D^fA_f)$ $(B^4F_bA^dE_9A^4E_8F^cB_c)$
$\overline{B^0}$: $(D_0F_8E_7D_b)$ $(D_7D_dE_0F_g)$ $F_fE_9E_fF_9$ $(B^8D_6C^8A_8A^bC_bB^2F_2)$ $(B^fF_5A^dC_dC^gA_gB^9D_1)$
$\overline{C^0}$: $(A_1B_gF_8F_1)$ $(F_gF_9B_1A_g)$ $D_1B_0D_gE_0$ $(D^1B_3F^gC_8B^1D_aC^1E_2)$ $(B^9D_7F^0C_9D^gB_eC^gE_f)$
$\overline{D^0}$: $(A_1E_2D_fA_f)$ $(E_fA_gA_2D_2)$ $E_0C_fB_0C_2$ $(C^1F_1A^fE_bD^fB_dE^1C_e)$ $(C^gF_gE^2C_3D^2B_4A^2E_6)$
$\overline{E^0}$: $(A_1C_5B_dA_f)$ $(A_dA_gB_1C_9)$ $C_0B_eB_0C_e$ $(E^4A_4F^dF_5D^fD_fE^1C_2)$ $(E^gC_cD^gD_gF^0F_9E^dA_a)$
$\overline{F^0}$: $(A_0C_1B_9F_0)$ $(E_9E_0A_1C_5)$ $C_dF_1C_0E_1$ $(E^0A_eF^9B_aF^8B_8C^0E_g)$ $(A^dD_dC^1E_2E^4A_4A^5D_5)$

and obtain the remaining $\overline{y^i}$ by uniform translations mod 17, for any $i \in \mathbf{Z}_{17}$.

Each vertex of $Y$ belongs exactly to twelve such copies $\mathcal{L}$ of $L(Q_3)$. Figure 2 shows the complements of $A_0$ in four of the twelve copies of $L(Q_3)$ containing

$A_0$ (sharing the long vertical edges), where the black vertices form the 4-cycles containing $A_0$, and some edges and vertices appear twice, in fact once per copy of $L(Q_3)$. For example, the leftmost and rightmost edges must be identified by parallel translation. Alternate internal anti-diagonal 2-paths in the figure also coincide with their directions reversed; (notice that the middle vertices of these 2-paths are the neighbors of $A_0$ in $\mathcal{S}$, and that their degree-one vertices are at distance 2 from $A_0$ in $\mathcal{S}$). The oriented 9-cycles of the $(\eta\vec{C}_9;\vec{P}_4)$-OAC of $\mathcal{S}$ in the proof of Theorem 2 intervene, as indicated on the figure, in the formation of the oriented 3-cycles (namely $E^0$, $E^3$, $D^0$, $C^0$) and copies of $L(Q_3)$ (namely $\overline{E^4}$, $\overline{D^2}$, $\overline{C^1}$, $\overline{F^0}$) induced respectively by the long vertical edges (namely $C_dC_1$, $A_3C_4$, $D_0B_2$, $E_1F_0$) and the 6-cycles they separate (namely $(C_dB_5A_3C_4A_5C_1)$, $(A_3E_4D_0B_2D_4C_4)$, $(D_0F_2E_1F_0F_aB_2)$, $(E_1E_9C_dC_1B_9F_0)$).

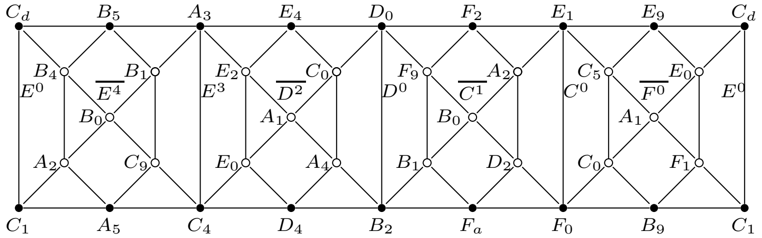

Figure 2: Complements of $A_0$ in four of the twelve copies of $L(Q_3)$

The symbolic information in Figure 2 can be set as in the following arrangement of four rows, followed by two similar arrangements that complete all the symbolic information provided by the copies of $K_4$ and $L(Q_3)$ that contain $A_0$:

$$\begin{array}{cccccccccccccccc}
E^0 & & \overline{E^4} & & E^3 & & \overline{D^2} & & D^0 & & \overline{C^1} & & C^0 & & \overline{F^0} & \\
C_d & & B_5 & & A_3 & & E_4 & & D_0 & & F_2 & & E_1 & & E_9 & \\
 & B_4 & & B_1 & & E_2 & & C_0 & & F_9 & & A_2 & & C_5 & & E_0 \\
 & & B_0 & & & & A_1 & & & & B_0 & & & & A_1 & \\
 & A_2 & & C_9 & & E_0 & & A_4 & & B_1 & & D_2 & & C_0 & & F_1 \\
C_1 & & A_5 & & C_4 & & D_4 & & B_2 & & F_a & & F_0 & & B_9 & \\
\hline
E^0 & & \overline{D^g} & & C^0 & & \overline{D^1} & & E^3 & & \overline{E^2} & & D^0 & & \overline{E^1} & \\
C_1 & & E_e & & E_g & & D_3 & & A_3 & & C_7 & & B_f & & C_6 & \\
 & D_1 & & A_f & & C_3 & & B_1 & & C_2 & & B_g & & B_e & & A_2 \\
 & & A_1 & & & & A_g & & & & A_1 & & & & A_g & \\
 & B_g & & C_e & & A_2 & & D_g & & A_f & & B_3 & & B_1 & & C_f \\
A_e & & D_e & & E_1 & & E_3 & & C_g & & C_b & & B_2 & & C_a & \\
\hline
E^0 & & \overline{D^f} & & D^0 & & \overline{C^g} & & C^0 & & \overline{F^g} & & E^3 & & \overline{E^g} & \\
A_e & & E_d & & D_0 & & F_f & & E_g & & E_8 & & C_4 & & B_c & \\
 & E_f & & C_0 & & F_8 & & A_f & & C_c & & E_0 & & B_d & & B_g \\
 & & A_g & & & & B_0 & & & & A_g & & & & B_0 & \\
 & E_0 & & A_d & & B_g & & D_f & & C_0 & & F_g & & A_f & & C_8 \\
C_d & & D_d & & B_f & & F_7 & & F_0 & & B_8 & & C_g & & A_c & \\
\hline
\end{array}$$

(Some edges are shared by two different of these three arrangements. In fact, each of the edges bordering the 2-paths mentioned above in anti-diagonal 4-paths is present also in the second or third arrangement. For example, the edge $B_1A_3$ of $E^4$ on the figure appears in the second arrangement).

The vertices $B_0$, $C_0$, $D_0$, $E_0$ and $F_0$ admit similar arrangements (see tables after proof of Theorem 4). Additions mod 17 yield the remaining information for neighboring copies of $K_4$ and $L(Q_3)$ at each vertex of $Y$.

**Theorem 4** *The graph $Y$ is an SF $(L(Q_3);K_3)$-UH graph. Moreover, each two copies of $L(Q_3)$ sharing a copy $H$ of $K_3$ in $Y$ also share $H$ with exactly one copy of $K_4$ in $Y$. Furthermore, each 4-cycle of $Y$ exists in just one copy of $L(Q_3)$ in $Y$. Thus, $Y$ is $\{K_4, L(Q_3)\}_{K_3}$-UH.*

*Proof.* Given a copy $H$ of $L(Q_3)$ in $Y$ and a copy $\Delta$ of $K_3$ in $H$, there exists a unique copy $\neq H$ of $L(Q_3)$ that shares with $H$ the subgraph $\Delta$. In addition, any edge of $H$ is shared by exactly three other copies $H' \neq H$, $H'' \neq H$ and $H''' \neq H$ of $L(Q_3)$. Because of the symmetry reigning between the copies of $K_4$ and of $L(Q_3)$ in $Y$, the statement follows. $\square$

Here is the data that must replace the symbols in Figure 2 to make explicit the complements of $y_0$ in the copies of $K_4$ and $L(Q_3)$ incident to $y_0$, for $y = A, B, C, D, E, F$, where rows are cited in parentheses and fourth rows cite each appearing vertex just once inside a pair of parenthesis preceded by $y_0$:

| | | |
|---|---|---|
| $(E^0\overline{E^4}E^3\overline{D^2}D^0\overline{C^1}C^0\overline{F^0})$<br>$(C_dB_5A_3E_4D_0F_2E_1E_9)$<br>$(B_4B_1E_2C_0F_9A_2C_5E_0)$<br>$A_0(B_0A_1)$<br>$(A_2C_9E_0A_4B_1D_2C_0F_1)$<br>$(C_1A_5C_4C_7B_2F_aF_0B_9)$ | $(E^0\overline{D^g}C^0\overline{D^1}E^3\overline{E^2}D^0\overline{E^1})$<br>$(C_1E_eE_gD_3A_3C_7B_fC_6)$<br>$(D_1A_fC_3B_1C_2B_gB_eA_2)$<br>$A_0(A_1A_g)$<br>$(B_gC_eA_2D_gA_fB_3B_1C_f)$<br>$(A_eD_eE_1E_3C_gC_bB_2C_a)$ | $(E^0\overline{D^f}D^0\overline{C^g}C^0\overline{F^g}E^3\overline{E^g})$<br>$(A_eE_dD_0F_fE_gE_8C_4B_c)$<br>$(E_fC_0F_8A_fC_cE_0B_dB_g)$<br>$A_0(A_gB_0)$<br>$(E_0A_dB_gD_fC_0F_gA_fC_8)$<br>$(C_dD_dB_fF_7F_0B_8C_gA_c)$ |
| $(F^0\overline{F^8}F^g\overline{A^4}D^2\overline{A^0}D^f\overline{A^d})$<br>$(F_9A_9C_8D_8D_2F_4B_dD_b)$<br>$(E_8C_dE_4F_0D_dC_4B_9D_0)$<br>$B_0(E_0C_0)$<br>$(C_4E_9D_0B_8C_dD_4F_0E_d)$<br>$(C_9A_8F_8D_6B_4F_dD_fD_9)$ | $(F^0\overline{E^0}D^f\overline{E^g}F^g\overline{E^3}D^2\overline{E^4})$<br>$(C_9C_eA_fA_cC_8C_3A_2A_5)$<br>$(A_dA_1C_gC_dA_4A_gC_1C_4)$<br>$B_0(C_0A_0)$<br>$(A_gC_5C_4A_eA_1C_cC_dA_3)$<br>$(B_1B_eB_dB_cB_gB_3B_4B_5)$ | $(F^0\overline{C^1}D^2\overline{D^0}D^f\overline{C^g}F^g\overline{C^0})$<br>$(B_1F_aD_2C_2A_fF_fF_8D_g)$<br>$(B_2F_0E_fA_1E_gD_0F_1A_g)$<br>$B_0(A_0E_0)$<br>$(D_0E_1A_gE_2F_0B_fA_1F_g)$<br>$(F_9F_2A_2C_fD_fF_7B_gD_1)$ |
| $(A^0\overline{A^4}F^8\overline{F^g}E^g\overline{E^3}E^4\overline{D^2})$<br>$(D_0D_6B_8F_gA_gC_3A_4B_2)$<br>$(F_8B_4C_gE_0A_2C_8D_4A_0)$<br>$C_0(B_0C_4)$<br>$(C_8D_2A_0E_8B_4B_gE_0A_3)$<br>$(E_4D_8F_0E_gC_cB_3A_1E_2)$ | $(A^0\overline{F^4}E^4\overline{E^8}F^8\overline{E^3}E^g\overline{E^4})$<br>$(E_4F_5C_5A_6B_8C_3A_dE_c)$<br>$(A_5B_dA_9B_4B_cC_9F_dC_8)$<br>$C_0(C_4C_d)$<br>$(C_9F_4C_8B_5B_dA_8B_4A_c)$<br>$(E_dE_5A_4A_7B_9B_3C_cF_c)$ | $(A^0\overline{D^f}E^g\overline{E^0}E^4\overline{F^0}F^8\overline{A^d})$<br>$(E_dE_fA_gB_eC_5F_fF_0D_9)$<br>$(A_eE_0B_1B_dE_9A_0D_fC_9)$<br>$C_0(C_dB_0)$<br>$(A_0D_dC_9A_fE_0C_1B_dF_9)$<br>$(D_0B_fA_dC_eA_1F_7B_9D_b)$ |
| $(D^0\overline{D^2}A^0\overline{C^1}B^8\overline{B^a}B^2\overline{C^1})$<br>$(A_0A_3E_4F_2F_8D_aF_2E_1)$<br>$(C_4E_2D_8B_0F_1D_4A_2F_0)$<br>$D_0(E_0D_2)$<br>$(D_4A_1F_0B_4E_2E_8B_0F_a)$<br>$(B_2A_4C_0F_aD_6E_aF_9B_1)$ | $(D^0\overline{A^f}B^2\overline{B^4}A^0\overline{B^6}B^8\overline{A^2})$<br>$(B_2C_bD_bF_3E_4F_5F_fB_6)$<br>$(F_bE_fF_dE_2E_6D_dC_fD_4)$<br>$D_0(D_2D_f)$<br>$(D_dC_2D_4E_bE_fF_4E_2F_6)$<br>$(B_fB_bF_2F_cE_dF_eD_6C_6)$ | $(D^0\overline{C^g}B^8\overline{B^0}B^2\overline{F^g}A^0\overline{D^f})$<br>$(B_fB_gF_8E_7D_bB_9C_0A_d)$<br>$(F_7B_0E_9E_fB_dF_0A_gD_d)$<br>$D_0(D_fE_0)$<br>$(F_0A_fD_dF_gB_0D_9E_fC_d)$<br>$(A_0E_gF_fD_7F_9C_9E_dA_e)$ |
| $(C^1\overline{F^1}C^g\overline{D^f}A^d\overline{A^0}A^4\overline{C^2})$<br>$(A_1C_2E_fA_eC_dF_dD_4A_4)$<br>$(A_fD_2E_dA_0B_4D_fB_2C_0)$<br>$E_0(B_0D_0)$<br>$(D_fA_2C_0B_fD_2B_dA_0E_4)$<br>$(E_2C_fA_gA_dD_dF_4C_4A_3)$ | $(C^1\overline{B^a}A^4\overline{B^8}C^g\overline{A^d}B^2\overline{C^1})$<br>$(E_2E_aE_8E_6E_fE_7E_9E_b)$<br>$(D_6F_9F_7D_2D_bF_8F_aD_f)$<br>$E_0(D_0F_0)$<br>$(F_8F_2D_fD_8F_9F_fD_2D_9)$<br>$(F_1D_aD_4F_6F_gD_7D_dF_b)$ | $(C^1\overline{F^0}A^d\overline{C^g}A^4\overline{F^g}C^g\overline{C^0})$<br>$(F_1C_1C_dA_9E_8C_cA_gD_g)$<br>$(B_9A_0C_8F_9E_gC_0B_1F_8)$<br>$E_0(F_0B_0)$<br>$(C_0E_1F_8C_9A_0B_8F_9B_g)$<br>$(A_1C_5E_9A_8C_4C_gF_gD_1)$ |
| $(C^0\overline{F^0}F^8\overline{A^d}B^0\overline{B^2}B^a\overline{C^0})$<br>$(A_0C_1B_9B_dD_fE_bF_aB_2)$<br>$(C_dF_1D_bB_0E_2E_9B_1D_0)$<br>$F_0(E_0F_9)$<br>$(E_9A_1D_0C_9F_1D_dB_0F_2)$<br>$(E_1D_eD_8B_aB_9A_7F_7D_5)$ | $(C^0\overline{B^1}B^a\overline{C^9}F^8\overline{C^8}B^0\overline{B^9})$<br>$(E_1D_eD_8B_aB_9A_7F_7D_5)$<br>$(E_aF_gA_8F_1D_7E_8D_gE_9)$<br>$F_0(F_9F_8)$<br>$(E_8D_1E_9D_aF_gA_9F_1E_7)$<br>$(E_gD_cF_aA_aB_8B_7D_9D_3)$ | $(C^0\overline{C^g}B^0\overline{F^g}B^a\overline{C^g}F^8\overline{F^g})$<br>$(E_gA_fD_fF_6D_8E_4C_0C_c)$<br>$(F_fB_0D_4F_gC_8D_0A_gE_8)$<br>$F_0(F_8E_0)$<br>$(D_0B_gE_8E_fB_0D_6F_gC_4)$<br>$(A_0B_fF_7E_6D_2B_4B_8C_g)$ |

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