Unboundedness for Generalized Odd Cyclic Transversality

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All graphs considered in this note are finite and without multiple edges and, unless the contrary is clear from the context, without loops. In 1962 (1984) P. Erdős and L. Pósa (V. Neumann-Lara) proved the following Conjecture(s) [1,3] ([4]):

For every positive integer $k$ there exists another positive integer $k'$ such that for every graph $G$, either

(i) $G$ has $k$ vertex-disjoint (long and/or even) cycles or

(ii) there exists a subset $X$ of less than $k'$ vertices of $G$ such that $G \setminus X$ has no (long and/or even) cycles.

By "long" cycles we mean cycles of length $\geq \mu$, where $\mu$ is a fixed positive integer. In fact, Neumann-Lara proved his cases by means of Menger and Ramsey Theorems. We assert that, when replacing in this Erdős-Pósa Theorem the term "cycle", then the modified conjecture becomes false. In fact, we have the following.

**Theorem 1.** Given an integer $k > 0$, there exists a graph $G$ without disjoint odd cycles such that the number of vertices of $G$ whose removal destroys all the odd cycles of $G$ is higher than $k$.

**Proof.** We use the following construction technique. Let $G_1$ be a graph with vertices $u_1, v_1, \ldots$; $G_2$ be a graph with vertices $u_2, v_2, \ldots$. By the weak direct

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product of $G_1$ and $G_2$, a graph on the Cartesian product of their vertex sets is understood such that e.g. $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent vertices in the product iff $u_1, v_1$ are adjacent in $G_1$ and $u_2, v_2$ are adjacent in $G_2$. We perform the weak direct product $W_{m,n}$ of an odd cycle $C_m$ of arbitrarily large odd size $m$ and a graph $Q_n$ formed by a path $P_n$ of arbitrarily large length $n$ and a loop at one of the ends of $P_n$. See Fig. 1. for an example of this, with point $O$ to be used later on.

The surface obtained by identifying in the rectangle $X = [0, m] \times [0, n]$ of the Euclidean plane $\mathbb{R}^2$ taken with standard Cartesian coordinates the points on its vertical sides $\{0\} \times [0, n]$ and $\{m\} \times [0, n]$ that have the same coordinates is a cylinder $Y$. If the vertices of $W_{m,n}$ are represented by the point of $Y$ having integer coordinates $(i, j)$ in $X$ with $0 \leq i \leq m = 2r + 1$ and $0 \leq j \leq n$, so that the points $(0, j)$ and $(2r + 1, j)$ represent the same vertex of $W_{m,n}$, for each integer $j$ in $[0, n]$, then the edges of $W_{m,n}$ can be represented as the line segments of the forms $(i, j)(i + 1, j - 1)$, $(i, j)(i - 1, j + 1)$ and $(0, j)(0, j + 1)$. As an example to this, see the rectangle $[0, 9] \times [0, 7]$ representation for $W_{9,7}$ in Fig. 2. The total angle of a directed cycle $C$ of $W_{5,3}$ in its representation in Fig. 1. around the center point $O$ is the sum of the angles spanned by the arcs of $C$ with signs attributed according to their orientation around $O$ (for example, + signs for angles of arcs
rotating clockwise around $O$). The same observation holds for any other $W_{m,n}$. By representing $W_{m,n}$ as in Fig. 2., we see that any odd cycle $C$ of $W_{m,n}$ has exactly an odd number of horizontal edges, i.e. edges of the form $(i,0)(i+1,0)$. Thus, by representing $W_{m,n}$ as in Fig. 1., we see that the image of any odd cycle necessarily covers a total angle around $O$ which is a nonzero odd multiple of $2\pi$ radians. In particular, every odd cycle $C$ of $W_{m,n}$ has vertices having all the possible values on their first coordinates. They have also an odd number of edges of the form $(0,i)(0,i+1)$. Note that the larger $m$ and $n$ are, the higher the cardinality $k'$ of a blocking set of vertices for all the odd cycles is. In fact, $k' = \min\{r+1, n+1\}$. (For example, $\{(2x,0) : x = 0, \ldots, m\}$ and $\{(0,y) : y = 0, \ldots, n\}$ are such blocking sets and no other such a blocking set may have more vertices than these). To see that each two odd cycles of $W_{m,n}$ do have a common vertex, we need the following lemma. For a positive integer $s$, we denote by $Z_s$ the factor ring of the ring of
integers mod $s$. The elements of $\mathbb{Z}_s$ are represented by the smallest nonnegative numbers in them.

Lemma 2. Let $s$ be an odd positive integer. Let $F$ be the family of functions $f$ from $\mathbb{Z}_s$ into the nonnegative integers such that either $|f(i + 1) - f(i)| = 1$ or $f(i + 1) = f(i) = 0$, for every $i \in \mathbb{Z}_s$. Then the number of pairs $\{i, i + 1\} \subseteq \mathbb{Z}_s$ such that $f(i) = f(i + 1) = 0$ is odd. Moreover, for every two members $f$ and $g$ of $F$, there exists $x \in \mathbb{Z}_s$ such that $f(x) = g(x)$.

Proof. A first consequence that we observe from the hypotheses of the statement is that $f(i) = f(i + 1) = 0$, for at least one $i \in \mathbb{Z}_s$. The first assertion of the Lemma becomes clear if we consider the parity of $i - f(i)$ so that $i$ takes the values $0, 1, 2, \ldots, s - 1$, and again $0$. With respect to the second assertion, assume, to the contrary, that there are $f$ and $g$ in $F$ which do not have a common value on any $x \in \mathbb{Z}_s$. We may assume that $f$ is not identically null, i.e. that there exists $x \in \mathbb{Z}_s$ such that $f(x) > 0$. Then there are elements $i$ and $i'$ of $\mathbb{Z}_s$ such that $f(i) = f(i') = 0$ and $f$ is positive at the elements of the sequence $J = \{i + 1, i + 2, \ldots, i' - 1\}$. Moreover, we may even assume that $g$ annihilates somewhere in $J$, since every function in $F$, in particular $g$, must be null somewhere in $\mathbb{Z}_s$, and $f$ and $g$ do not have common values. Furthermore, $i$ and $i'$ have the same parity, meaning that $i - i' \in \mathbb{Z}_s$ has an even representative in $[0, s)$. The fact that $f$ and $g$ differ in particular between $i$ and $i'$ implies easily that the first element $j \in \mathbb{Z}_s$
in \( J \) satisfying \( g(j) = 0 \) must differ in parity from \( i \) and \( i' \), that is \( j - i \) and \( i' - j \) have odd representatives in \([0, s]\). (See Fig. 3. for an illustration of this situation, where exemplified portions of \( f \) and \( g \) are represented by means of piecewise-linear extensions). Similarly, the last element \( j' \in Z_s \) in the above sequence satisfying \( g(j') = 0 \) must again differ in parity with \( i \) and \( i' \). It is easy to conclude from this that the number \( \alpha(J) \) of elements \( x \in J \) such that \( g(x) = g(x + 1) = 0 \) is even. (For example, in Fig. 3., \( \alpha(J) = 2 \), since the subset of elements \( x \) as in the preceding sentence is \( \{i + 3, i + 6\} \)). Finally, let \( J_1, J_2, \ldots, J_\mu \) be the collection of all subsets of \( Z_s \) which can be obtained in the same way that \( J \) was obtained in the above argument. Then \( \alpha(J_1) + \alpha(J_2) + \ldots + \alpha(J_\mu) \) is even, contradicting the first assertion of the Lemma. ■

Proof. (of Theorem 1, continuation) If there were two disjoint odd cycles \( C \) and \( D \) of \( W_{m,n} \), we might assume that \( C \) and \( D \) are directed with total angles around \( O \) having the same signs. (They have a sign since they are nonzero multiples of \( 2\pi \)). It can be seen that under these assumptions there would be closed walks \( W_C = (u_0, u_1, \ldots, u_s) \) and \( W_D = (v_0, v_1, \ldots, v_s) \) of odd length, covering \( C \) and \( D \) respectively, with the total angle covered by \( W_C \) (\( W_D \)) around \( O \) equal to the larger of the total angles covered by \( C \) end \( D \) and such that \( \pi_1(u_i) = \pi_1(v_i) \) for every \( i = 0, \ldots, s \), where, for every vertex \( v \) of \( W_{m,n} \), \( \pi_1(v) \) is the projection of \( v \) onto the \( j^{th} \) coordinate of its representation in \( \mathbb{X} \) (\( j = 1, 2 \)). The associated integer sequences \( \pi_2(W_C) \) and \( \pi_2(W_D) \) would fulfill the hypotheses of Lemma 2, and this takes us to conclude that \( C \) and \( D \) have a vertex in common. ■

The construction given in the proof of Theorem 1 can be adapted to the following extension. Given integers \( L \) and \( N \) such that \( 0 < L < N \), we say that a cycle of length congruent to \( L \) modulo \( N \) is an \( LN \)-cycle. We also say that an \( LN \)-cycle is a generalized odd cycle.

**Theorem 3.** Let \( L = 2^p q \) and \( N = 2^r s \), where \( p, q, r \) and \( s \) are integers, \( q \) and \( s \) are odd > 0 and \( 0 < p < r \). Given a positive integer \( k \), there exists a graph without disjoint \( LN \)-cycles for which the number of vertices whose removal destroys all \( LN \)-cycles is higher than \( k \).

Remark. This Theorem provides an alternative way to a construction satisfying the claim of Theorem 1.

Proof. In the representation of \( W_{m,n} \) given as in Fig. 2., replace:

(i) each horizontal edge (with ends in \( (\pi_2)^{-1}(0) \)) by an \( L \)-path.

(ii) each nonhorizontal (or diagonal) edge by an \( N \)-path.

This modification yields a graph \( NW_{m,n} \) satisfying the contention of the Theorem 3. In fact, every \( LN \)-cycle \( C \) under the hypotheses of the Theorem covers a total angle around a point \( O \) in the center of a representation of \( NW_{m,n} \) as in Fig. 1., which is a nonzero multiple of \( 2\pi \). Thus Lemma 2 can be applied under the present hypotheses in a similar way as it was applied in the proof of Theorem 1.
Remark. The fact, observed in the preceding proof, that every \( L_N \)-cycle \( C \) under the hypothesis covers a nonzero multiple of \( 2\pi \) as a total angle around \( O \), is not the case if for example we take \( N = 3 \) and \( L = 1 \) for there are disjoint 13-cycles in \( ^{3,1}W_{m,n} \) \( (m \text{ odd and } n, \text{ both large} ) \) formed by the horizontal copy of \( P_4 \) and two diagonal copies of \( P_5 \) each.

**Question 4.** Decide whether the following question, whose answer in the affirmative is analogous to the Erdős-Pósa Theorem, is valid or not: given integers \( L \) and \( N \) such that \( 0 < L < N \) but not of the form prescribed in Theorem 3, does there exists, for every positive integer \( k \), another positive integer \( k' \) such that for every graph \( G \), either

(i) \( G \) has \( k \) vertex-disjoint \( L_N \)-cycles or

(ii) there exists a subset \( X \) of less than \( k' \) vertices of \( G \) such that \( G \setminus X \) has no \( L_N \)-cycles? In particular, is this the case for every \( O_N \)-cycles?

**Remark.** Recently, C. Thomassen [6] derived a general sufficient condition for a family \( F \) of graphs to have the Erdős-Pósa property that for every natural number \( k \) asserts there is another one \( k' \) such that if \( G \) is a graph with \( S \subseteq V(G) \) and \( |S| \leq k' \) and a subgraph of \( G - S \) is a member of \( F \) then \( G \) has \( k \) pairwise disjoint subgraphs which are members of \( F \). Clearly, the (Neumann–Lara) Erdős-Pósa Theorem mentioned above may be restated by saying that the family \( F \) of all (long and/or even) cycles has the Erdős-Pósa property. In particular, Thomassen proved that for any fixed positive integer \( N \), the collection of \( O_N \)-cycles has the Erdős-Pósa property, thus establishing Question 4 for \( O_N \)-cycles. Also, [6] provides an alternative construction leading to counter-examples as in Theorem 1 above.

We consider the adaptation of a Conjecture attributed to T. Gallai and D. H. Younger, Prob. 47 in [2, page 252], to odd cycles.

**Conjecture 5.** Given a positive integer \( n \), there exists a least integer \( f(n) \) such that in any digraph with at most \( n \) arc-disjoint (odd) directed cycles there are \( f(n) \) arcs whose deletion destroys all (odd) directed cycles.

**Theorem 6.** Conjecture 5 is false.

**Proof.** In order to disprove Conjecture 5, the following modification on the graph \( V_{m,n} \) should be considered: Replace each of the interior vertices of \( V_{m,n} \), i.e. those vertices having first coordinates different from zero, by a 6-cycle and orient the edges as arcs, as indicated in Fig. 4. A modification of the argument used in proving Theorem 1 shows that this modified version \( V_{m,n} \) of \( W_{m,n} \) is sufficient to disprove Conjecture 5 for \( n = 1 \). □

**Theorem 7.** The analogue of Conjecture 5 for the \( L_N \)-cycles considered in Theorem 3 is false.

**Proof.** It is enough to replace in the graphs \( V_{m,n} \) the bottom horizontal edges by paths of length \( L \) as indicated in the proof of Theorem 3 and every other edge by a path of length \( N \). □
Remark. Recently we learned that the characterization of all graphs without disjoint odd cycles is being planned in a joint paper by A. Gerards, L. Lovász, P. Seymour and A. Schrijver into three types of graphs, by means of P. Seymour's characterization of graphical unimodular matroids [5]. In fact, they claim that each such a graph falls into one of the following three cases:

(i) Complete bipartite graphs to which we add an extra vertex joined to all vertices in both vertex parts.
(ii) Complete graphs with a triangle added on one of the vertex parts.
(iii) Graphs embeddable in the complex plane represented as a disk, the pairs of antipodal points of whose boundary are identified, so that an odd cycle spans this boundary and such that all disk interior cycles are even.

We remark that the examples provided by Theorem 1 above or by Theorem 3 for $L = 1$ and $N = 2$ fall into type (iii).

Remark. After the first version of this paper was finished, C. Thomassen suggested the following alternative construction for graphs fulfilling the claims of Theorem
3. Consider the grid represented in Fig. 5., where numeric labels assigned to some vertices will be used subsequently. Subdivide all the edges of the grid in such a way that they all become paths of length $N$. Then add a new path of length $L$ whose ends are the two vertices labelled with $j$, for every $j = 1, 2, \ldots, k$. The resulting graph may be used to give an alternative proof of Theorem 3. Unfortunately, this construction does not reduce the gap between Thomassen's results in [6] and Theorem 3 above. Thus, we are at present inclined to believe that $1_3$-cycles could satisfy the Erdős-Pósa property. On the other hand, the Thomassen construction just described can be adapted to a construction leading to Theorem 6 or 7.

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